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Hopf symmetry breaking and confinement in (2+1)-dimensional gauge theory

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Abstract: Gauge theories in 2+1 dimensions whose gauge symmetry is spontaneously broken to a finite group enjoy a quantum group symmetry which includes the residual gauge symmetry. This symmetry provides a framework in which fundamental excitations (electric charges) and topological excitations (magnetic fluxes) can be treated on an equal footing. In order to study symmetry breaking by both electric and magnetic condensates we develop a theory of symmetry breaking which is applicable to models whose symmetry is described by a quantum group (quasitriangular Hopf algebra). Using this general framework we investigate the symmetry breaking and confinement phenomena which occur in (2+1)-dimensional gauge theories. Confinement of particles is linked to the formation of string-like defects. Symmetry breaking by an electric condensate leads to magnetic confinement and vice-versa. We illustrate the general formalism with examples where the symmetry is broken by electric, magnetic and dyonic condensates.

Keywords: Topological Field Theories, Spontaneous Symmetry Breaking, Confinement, Quantum Groups.

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1. Introduction

One of the roads towards an understanding of confinement starts with the proposal of 't Hooft and Mandelstam [1, 2] to think of it in terms of the breaking of a dual or magnetic symmetry by a condensate of magnetic monopoles or, in 2+1 dimensions, vortices. While this idea has been very fruitful, it has not yet led to a rigorous proof of confinement. One reason for this is that the supposed magnetic symmetry is not manifest in the usual formulation of gauge theory. It is therefore difficult to study its breaking in detail. Other approaches also try to link the phenomenon of electric confinement to condensate physics in the magnetic sector, but there is no general consensus as to which magnetic excitations should be the ones to condense. A review which discusses the progress made recently on the most promising confinement scenarios, emphasizing the role of magnetic symmetry, is [3].

In this paper, we will study the Higgs and confinement transitions in a class of theories where both the electric and the magnetic symmetry are manifest and where we have a clear picture of the possible magnetic excitations. The theories in question are (2+1)-dimensional gauge theories in which the gauge group G is broken down to a discrete group H. The full electric-magnetic symmetry in such "discrete gauge theories" is described by a quantum group or quasitriangular Hopf algebra: the quantum double $D(H)$ of the discrete unbroken gauge group. If H is abelian, then $D(H)$ is just the group algebra of the group $\tilde{H} \times H$, so that we have the electric group H and a dual magnetic group $\tilde{H} \cong H$. However, if H is non-abelian, then $D(H)$ is not a group algebra and so the total symmetry is not described by a group. As a consequence, the discussion of $D(H)$-symmetry breaking cannot proceed in the usual way when H is non-abelian; we need to generalize the concepts involved in the discussion of symmetry breaking so that we can study symmetry breaking not only for symmetries described by groups, but also for symmetries described by quantum groups. A large part of this paper (sections 6 and 7) is thus devoted to setting up a formalism for the description of symmetry breaking and confinement in theories with a quantum group symmetry. The procedure that emerges has two steps: the original symmetry algebra is first broken down to a residual Hopf algebra and this algebra is subsequently projected onto a Hopf algebra

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which classifies the unconfined excitations. This formalism can in principle be applied to any planar system with a quantum group symmetry and could thus have applications in physics ranging from (2+1)-dimensional quantum gravity to the quantum Hall effect.

In the rest of the paper, we apply our formalism to discrete gauge theories. The results on discrete gauge theories that we have obtained include descriptions of

- electric gauge symmetry breaking and the corresponding magnetic confinement,
- symmetry breaking by manifestly gauge invariant magnetic condensates and the ensuing electric confinement,
- symmetry breaking and confinement by various other types of condensates, such as non-gauge invariant magnetic condensates and dyonic condensates.

Some of these results have been announced already in [4]. The present paper gives the full story, including derivations and explicitly worked examples. These include a complete treatment of the discrete gauge theories whose gauge group is an odd dihedral group.

The detailed setup of the paper is as follows: In section 2, we give a quick introduction to discrete gauge theories. We describe the fundamental and topological excitations and the topological interactions between these excitations, which play an important role in the confinement discussion. At the end of this section, we also give a non-technical preview of the effects of symmetry breaking by a condensate of fundamental, electrically charged particles. In section 3, we review some basic notions from the theory of quantum groups and establish notation. In section 4, we describe the quantum groups which reproduce the spectrum and interactions of discrete gauge theories: the quantum doubles of finite groups. Some examples of quantum doubles which we use throughout the paper are introduced in section 5. In section 6, we develop a general method for the study of spontaneous symmetry breaking in systems with a quantum group symmetry and in section 7, we give a general discussion of the confinement phenomena that accompany this symmetry breaking. Section 8 gives some motivation for the choice of the specific condensates we study in the rest of the paper. In section 9, we discuss the phases of discrete gauge theories that occur when the electric symmetry is broken by condensation of an electrically charged particle that does not carry magnetic flux. In section 10, we discuss the phases that occur when the magnetic symmetry is broken by a gauge invariant magnetic condensate. In section 11, we discuss the simultaneous breaking of the electric and magnetic symmetries as a consequence of the condensation of a pure, non gauge invariant, magnetic flux. In section 12, we present some results on dyonic condensates. Finally, in section 13, we give a brief summary and an outlook.

2. Discrete gauge theories: physical setting

We use the term discrete gauge theory for a (2+1)-dimensional Yang-Mills-Higgs theory in which the Higgs field has broken the (continuous) gauge group G down to a finite group H.[1]

[1] Examples of renormalisable Higgs potentials which break SO(3) down to various discrete subgroups are given in [5].

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Such theories are discussed in great detail in [6]–[11] and we will only give a brief account of some of the results in those papers here, omitting derivations. As a consequence of the symmetry breaking, discrete gauge theories contain topological defects which are labeled by elements of $\pi_1(G/H)$. By the exact homotopy sequence, this corresponds to $\pi_0(H) = H$ when $\pi_1(G)$ is trivial. It follows that, when G is simply connected, the defects are characterized by elements of the unbroken group H.[2] The element of H that characterizes a defect may be identified as the value of an untraced Wilson loop integral around the defect. In analogy with electromagnetism, we will call the value of this loop integral the "magnetic flux" through the loop and we will call the defects magnetic fluxes. In this setting, fluxes are thus group valued. It is clear that the values of the Wilson loop integrals are indeed elements of the unbroken group H, since, if they were not, parallel transport around the closed loop would not leave the Higgs field's expectation value invariant.

Since the Wilson loop is not traced, there is a non-trivial action of the unbroken group H on fluxes, given by conjugation: a flux $g \in H$ is sent to hgh^{-1} by the element h of H. This transformation rule is obtained by applying a gauge tranformation with gauge function equal to h at the base point of the Wilson loop. As a consequence, the fluxes are organized into gauge multiplets, one for each conjugacy class of H. Thus, the distinct types of flux-carrying particle are labeled by the conjugacy classes $A, B, \ldots$ of H, while a particle of, say, type A, has an internal Hilbert space of dimension equal to the number of elements of the conjugacy class A.

Apart from the topological fluxes, we also allow fundamental charged particles which are labeled by the irreducible representations of the unbroken group H. The internal Hilbert space of a particle that carries the irrep α of H is just the module V_α of α and the action of the gauge group on this space is just the action given by the matrices of the irrep α. In addition to charges and fluxes, one has dyons, particles which carry both flux and charge. The charges of dyons with flux $g \in H$ are characterised by the irreducible representations of the centraliser associated with the conjugacy class of g in H. In other words, the charge of a dyon is characterised by a representation of the subgroup of the gauge group that leaves the flux of the dyon invariant. The set of electric charges available to a dyon thus depends on the flux of the dyon, indicating that there must be a non-trivial interplay between the electric and magnetic symmetries in this theory.

Now that we have given the natural set of quantum numbers labeling the different sectors (or particle charges) in this theory, let us turn to the interactions. Because the unbroken group is discrete, the gauge fields of the theory are massive, with mass proportional to the length v of the vacuum expectation value of the Higgs field. As a consequence, the electric and magnetic gauge interactions are screened with a screening length inversely proportional to v. We are interested in the low energy or long distance limit of these theories, or equivalently in the limit in which the expectation value of the Higgs field becomes large. In this limit, the theory becomes topological; the only interacions between the particles that survive are ultra-short range interactions, that may be described by fusion rules, and

[2] One may argue that even in a theory where G is not simply connected, the set of *stable* fluxes will still be labeled by elements of H, due to the presence of (Dirac) magnetic monopoles in the three-dimensional theory that underlies the two-dimensional theory we are considering here [11].

non-local Aharonov-Bohm interactions (the Aharonov-Bohm effect is not screened by the Higgs effect [12, 13, 9, 11]). These Aharonov-Bohm interactions may be described by the action of a (coloured) braid group on the multi-particle states involved and hence we will refer to it as "braiding"

The fusion rules for charges are given by the tensor product decomposition of H-irreps. The fusion product of two fluxes is found by concatenation of the associated Wilson loops, which leads to the conclusion that the fusion product of fluxes $g_1, g_2 \in H$ is the flux labeled by $g_1 g_2 \in H$. The effect of braiding a charge α around a flux $g \in H$ is given by the action of g on V_α. If α is a one-dimensional irrep of H, then this is the usual Aharonov-Bohm phase factor, but if α is higher-dimensional, then the action of g on V_α will be described by the matrix $\alpha(g)$, which will not necessarily reduce to a phase factor. It follows that, if the unbroken group is non-abelian, discrete gauge theories allow for non-abelian braiding between charges and fluxes; if a charge is first taken around a flux g_1 and then around a flux g_2, then the effect on the wave function of the charge may be different than if it is taken first around g_2 and then around g_1, simply because one may have $\alpha(g_1)\alpha(g_2) \neq \alpha(g_2)\alpha(g_1)$. The braiding between fluxes may be found by contour manipulation. If a particle with flux g_1 is taken around a particle with flux g_2, then its flux will change from g_1 to $g_2 g_1 g_2^{-1}$, i.e. braiding between fluxes is given by conjugation. Braiding and fusion of dyons will be described in section 4.

Much of the rest of this paper is devoted to the development of a mathematical formalism which will help us describe the phases that result when the symmetry of a discrete gauge theory is broken by the formation of a condensate. However, in the special case where the condensate is purely electric, one may already get a fairly accurate picture of what happens using only the information in this section. The reason for this is that the formation of an electric condensate can be viewed as a simple modification of the Higgs condensate which broke G down to H in the first place. After this modification, the new residual gauge group will be the subgroup N of H which leaves the new condensate invariant. The spectrum of free excitations should thus consist of fluxes labeled by conjugacy classes of N, charges labeled by irreps of N and dyons with flux and centralizer charge. In short, everything should be as it was before, except that the role of H has been taken over by N. Any fluxes $h \in H \backslash N$ that were present when the new condensate formed, will pull a string. That is, their presence makes it impossible for the new Higgs condensate to be single valued and hence causes the expectation value of the Higgs field to develop a line-like discontinuity. The energy associated with this discontinuity will grow linearly with its length and as a consequence, the fluxes h outside N will be confined in "hadrons" whose overall flux does lie in N. The line discontinuities themselves can be viewed as domain walls between regions with different values v_1, v_2 of the Higgs expectation value. They may thus be characterized by an element $h \in H$ such that $hv_1 = v_2$, but this characterization is not unique, since hnv_1 will also equal v_2 for any $n \in N$. Therefore, the strings (or walls) should be characterised by a coset hN in H/N, or more precisely by a gauge orbit of such cosets. This picture of what happens when a purely electric condensate is formed is an important part of the intuition that will be used in the rest of the paper and it is a non-trivial test of the formalism we will develop that it must repoduce this picture.

3. Hopf symmetry in (2+1) dimensions

The spectrum, fusion and braiding that we have sketched in section 2, and also the fusion and braiding of dyons, may be derived in a purely algebraic way, using a quantum group symmetry that is present in the theory. The quantum group in question is the quantum double $D(H)$ of the unbroken group H. A description of this quantum group and its role in discrete gauge theory appears in section 4, followed by some concrete examples in section 5. The aim of the present section is to quickly remind the reader of some aspects of Hopf algebra theory and its application to the description of many particle systems that are relevant to us. We also establish notation and collect some formulae for reference. For much more information on Hopf algebras and quantum groups, one may consult for example [14]–[18].

A Hopf algebra is an associative algebra $\mathcal{A}$ with multiplication μ and unit 1, that has extra structures called counit, antipode and coproduct. These defining structures of a Hopf algebra guarantee that the spectrum of irreducible representations of a Hopf algebra has properties that mimic those of the particle spectrum of a physical theory.

The *coproduct* Δ is an algebra map from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A}$ with the following property, called coassociativity:

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta\,. \tag{3.1}$$

Here, id is the identity map on $\mathcal{A}$. Given two representations π^1, π^2 of the quantum group, the coproduct makes it possible to define a tensor product representation $\pi^1 \otimes \pi^2$ by the formula

$$\pi^1 \otimes \pi^2 : x \to \left(\pi^1 \otimes \pi^2\right)(\Delta(x))\,. \tag{3.2}$$

The coassociativity of Δ ensures that the tensor product of representations is associative. If we have a theory where the different particles carry irreducible representations of $\mathcal{A}$, then Δ provides us with a way of describing the action of $\mathcal{A}$ on multi-particle states. The decomposition of the tensor product representations defined by means of Δ will describe the fusion rules of the theory.

The *counit* ϵ of $\mathcal{A}$ is an algebra map from $\mathcal{A}$ to $\mathbb{C}$, or equivalently, a one-dimensional representation of $\mathcal{A}$, which satisfies

$$(\epsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}\,. \tag{3.3}$$

It follows that $\epsilon \otimes \pi \cong \pi \otimes \epsilon \cong \pi$ for any representation π of the quantum group. Thus, if we assume that the vacuum (or an $\mathcal{A}$-neutral particle) transforms in the representation ϵ of $\mathcal{A}$, then we get the fusion properties that one would expect. In other words, ϵ provides us with a vacuum representation of the algebra.

The antipode of $\mathcal{A}$ is a linear algebra antihomomorphism $S : \mathcal{A} \to \mathcal{A}$ which satisfies

$$\mu(S \otimes \mathrm{id})\Delta(a) = \mu(\mathrm{id} \otimes S)\Delta(a) = \epsilon(a)1 \tag{3.4}$$

for each $a \in \mathcal{A}$. If we are given a representation π of $\mathcal{A}$, then the antipode makes it possible to define the representation $\bar{\pi}$ conjugate to π by the formula

$$\bar{\pi}(a) = (\pi(S(a)))^{\mathrm{t}}\,, \tag{3.5}$$

where the t denotes matrix transposition. The properties (3.4) then ensure that the tensor product representations $\pi \otimes \bar{\pi}$ and $\bar{\pi} \otimes \pi$ will contain the trivial representation ϵ in their decomposition. Thus, a particle which carries the representation π and its antiparticle, which carries the representation $\bar{\pi}$ may indeed annihilate.

We now briefly recall the notion of a Hopf subalgebra and of the dual of a Hopf algebra, as they will turn out to be important in our discussion of Hopf symmetry breaking.

Definition 1. *A* Hopf subalgebra *of a Hopf algebra $\mathcal{A}$ is a subalgebra $\mathcal{B}$ of $\mathcal{A}$ which satisfies*

$$1 \in \mathcal{B}\,, \qquad S(\mathcal{B}) \subset \mathcal{B}\,, \qquad \Delta(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{B}\,. \tag{3.6}$$

This implies that $\mathcal{B}$ is itself a Hopf algebra, with "the same" structures as $\mathcal{A}$.

Definition 2. *For any finite dimensional Hopf algebra $\mathcal{A}$, the* dual Hopf algebra *is the vector space $\mathcal{A}^*$ of linear functionals from $\mathcal{A}$ to $\mathbb{C}$ with Hopf algebra structures $1^*, \mu^*, \Delta^*, \epsilon^*$ and S^* defined by:*

$$\begin{array}{rlrl} 1^* : & a \mapsto \epsilon(a) & \mu^*(f_1, f_2) : & a \mapsto f_1 \otimes f_2 \circ \Delta(a) \\ \epsilon^* : & f \mapsto f(1) & \Delta^*(f) : & (a_1, a_2) \mapsto f(\mu(a_1, a_2)) \\ S^*(f) : & a \mapsto f(S(a)) & & \end{array} \tag{3.7}$$

Here, a, a_1, a_2 are arbitrary elements of $\mathcal{A}$ and f, f_1, f_2 are arbitrary elements of $\mathcal{A}^*$.

In this paper, we will deal only with finite-dimensional semisimple Hopf algebras. We call Hopf algebra $\mathcal{A}$ semisimple if all $\mathcal{A}$-modules are fully reducible. A simple and important example of a finite dimensional semisimple Hopf algebra is the group algebra $\mathbb{C}H$ of a finite group H. Its comultiplication, antipode and counit are given on the basis of group elements $h \in H$ by the formulae

$$\Delta(h) = h \otimes h \qquad S(h) = h^{-1} \qquad \epsilon(h) = 1\,. \tag{3.8}$$

Clearly, we may similarly associate a Hopf algebra to any group. Thus, if we can describe the excitation spectrum and fusion properties of a physical system by means of a group then we may also describe them by means of a Hopf algebra. On the other hand, the converse is not necessarily true; there are physical systems whose symmetry algebra is a Hopf algebra, but not a group algebra. In fact, most systems which can be described by means of two-dimensional conformal field theory have this property (for reviews, see for instance [19, 20, 21]). Examples in (2+1) dimensions are the discrete gauge theories we will treat in this paper, but also (2+1)-dimensional gravity [22, 23] and certain fractional quantum Hall systems [24, 25].

In two spatial dimensions, the exchanges of a system of n particles are governed by the braid group B_n. There is a special class of Hopf algebras, called quasitriangular Hopf algebras or quantum groups, whose extra structure makes it possible to include a description of the braiding between particles in the Hopf algebraic framework. The extra feature

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which makes a Hopf algebra $\mathcal{A}$ quasitriangular is an invertible element $R \in \mathcal{A} \otimes \mathcal{A}$ which is called the universal R-matrix and which satisfies the following equations

$$\Delta^{\mathrm{op}} R = R\Delta \tag{3.9}$$

$$(\Delta \otimes 1)R = R_{13}R_{23} \tag{3.10}$$

$$(1 \otimes \Delta)R = R_{13}R_{12}\,. \tag{3.11}$$

Here, Δ^{op} is the comultiplication, followed by an exchange of the tensor factors in $\mathcal{A} \otimes \mathcal{A}$ and R_{ij} is an abbreviation for the action of R on the factors i and j of $\mathcal{A}^{\otimes 3}$, so for example $R_{12} = R \otimes 1$.

The R-matrix is used to describe braiding in the following way. If we have a system of n identical particles that transform in the representation π of $\mathcal{A}$, then the corresponding (internal) Hilbert space is the n-fold tensor product $(V_\pi)^{\otimes n}$, where V_π is the module of π. We can now exchange adjacent particles numbered i and $i+1$ by letting $R_{i,i+1}$ act on the state of the system and then flipping the i^{th} and $(i+1)^{th}$ tensor factors. We will call the map which flips the factors of a tensor product σ.

The defining properties of the R-matrix are chosen so that they make sure that exchanging particles by means of σR makes physical sense. The properties (3.10) and (3.11) make sure that braiding of two particles around a third one and then fusing them together gives the same result as fusing the two particles first and then braiding the result around the third one. The property (3.9) ensures that the exchanges commute with the action of the quantum group. Using either (3.9) and (3.10) or (3.9) and (3.11), one may also prove that in any representation, we have

$$(\sigma R \otimes 1)(1 \otimes \sigma R)(\sigma R \otimes 1) = (1 \otimes \sigma R)(\sigma R \otimes 1)(1 \otimes \sigma R)\,. \tag{3.12}$$

It follows that, for a system of n identical particles that carry a representation of the quantum group, the exchanges of adjacent particles, as performed using σR, satisfy the relations of the braid group. Hence, since R is invertible, they generate a representation of this group. Since the exchanges commute with the action of the quantum group $\mathcal{A}$, it follows that the system carries a representation of $\mathcal{A} \times B_N$.

Sometimes, a description of the spin of the particles in a two dimensional theory can also be incorporated into the Hopf algebraic description of the system. This is the case when the quasitriangular Hopf algebra that describes the system has an invertible central element c that satisfies the equations

$$\begin{aligned} c^2 &= uS(u)\,, \qquad S(c) = c\,, \qquad \epsilon(c) = 1 \\ \Delta(c) &= (R_{21}R_{12})^{-1}(c \otimes c)\,, \end{aligned} \tag{3.13}$$

where $u = \mu(S \otimes \mathrm{id})(R_{21})$. The element c is called the *ribbon element* and a quasitriangular Hopf algebra that has a ribbon element is called a *ribbon Hopf algebra.* The action of the ribbon element on the physical Hilbert space is interpreted as the action of a rotation of the system over 2π. In particular, the action of c on an irrep of $\mathcal{A}$ describes the effect of rotating the particle that carries this irrep. Because c is central, the action of c on an irrep may always be described by a scalar factor, which is called the spin factor of the irrep and of the corresponding particle. The equations that c has to satisfy make sure that the spin of the vacuum is trivial, that the spin of a particle and its antiparticle are equal and that rotating a system of two particles over an angle of 2π may be accomplished both by acting with c on the two-particle system (making use of Δ) and through braiding the two particles around each other and then rotating them separately.

4. The quantum double of a finite group

4.1 The double and its dual

The ribbon Hopf algebra that describes the fusion and braiding of the discrete gauge theory with unbroken group H is the quantum double $D(H)$ of H. As a vector space, $D(H)$ is $F(H)\otimes\mathbb{C}H$, the tensor product of the group algebra $\mathbb{C}H$ of H and its dual, the space $F(H)$ of functions on H. Since H is finite, we may identify this vector space with $F(H\times H)$, the space of functions on $H\times H$, and we may write elements of the double as such functions. On the double, we have the usual structures of a Hopf algebra: a multiplication $\bullet$, identity 1, comultiplication Δ, counit ϵ and antipode S:

$$\begin{aligned}
1(x,y) &:= \delta_e(y) \\
(f_1\bullet f_2)(x,y) &:= \int_H f_1(x,z)\, f_2(z^{-1}xz, z^{-1}y)\, dz \\
\epsilon(f) &:= \int_H f(e,y)\, dy \\
(\Delta f)(x_1,y_1;x_2,y_2) &:= f(x_1x_2,y_1)\,\delta_{y_1}(y_2) \\
(Sf)(x,y) &:= f(y^{-1}x^{-1}y, y^{-1})\,.
\end{aligned} \tag{4.1}$$

Here, the integrals over H are a convenient notation for the sum over all elements of H. We see that $D(H)$ is generated as an algebra by the elements $1\otimes g$ ($g\in H$) and $\delta_g\otimes e$ ($g\in H$). The elements $1\otimes g$ together form the gauge group H, while the elements $\delta_g\otimes e$ are a basis of $F(H)$ and can be interpreted as projections on the set of states with flux g in the theory. Both multiplication and comultiplication of the double are consistent with this interpretation. The universal R-matrix of $D(H)$ is given by the formula

$$R(x_1,y_1;x_2,y_2) = \delta_e(y_1)\delta_e(x_1y_2^{-1}) \tag{4.2}$$

and the ribbon element c is given by

$$c(x,y) = \bullet\circ(S\otimes \mathrm{id})(R_{21}) = \delta_e(xy)\,. \tag{4.3}$$

The dual $D(H)^*$ of $D(H)$ is $\mathbb{C}H\otimes F(H)$ as a vector space. This space may again be identified with $F(H\times H)$, so that we may realize both the structures of $D(H)$ and those of $D(H)^*$ on this space. The multiplication $\star$, unit 1^*, comultiplication Δ^*, counit ϵ^* and antipode S^* of $D(H)^*$ are given by

$$1^*(x,y) := \delta_e(x)$$

$$
\begin{aligned}
(f_1 \star f_2)(x,y) &:= \int_H f_1(z,y)\, f_2(z^{-1}x,y)\, dz \\
\epsilon^*(f) &:= \int_H f(x,e)\, dx \\
(\Delta^* f)(x_1,y_1;x_2,y_2) &:= f(x_1,y_1y_2)\, \delta_{x_2}(y_1^{-1}x_1y_1) \\
(S^* f)(x,y) &:= f(y^{-1}x^{-1}y,y^{-1})\,.
\end{aligned}
\tag{4.4}
$$

$D(H)$ and $D(H)^*$ have a canonical hermitean inner product, given by the same formula for both $D(H)$ and $D(H)^*$:

$$
(f_1,f_2) = \int_H \int_H f_1(x,y)\, \overline{f_2(x,y)}\, dx\, dy\,. \tag{4.5}
$$

The matrix elements of the irreps of both $D(H)$ and $D(H)^*$ are orthogonal with respect to this inner product. This follows from the theory of Woronowicz [26] for compact matrix quantum groups, which holds both for $D(H)$ and $D(H)^*$ [27], but it may also be proved directly.

4.2 Irreducible representations

The irreducible representations π_α^A of $D(H)$ have been classified in [28], using the fact that the double is a based ring in the sense of Lusztig [29]. An alternative way to classify the irreps of quantum doubles of groups makes use of the fact that $D(H)$ is a transformation group algebra [30]. Since we will be making rather extensive use of transformation group algebras in the sequel, we will follow this path. We follow the notation and conventions of [30]. First, we give a simplified definition of a transformation group algebra, adjusted to our needs, which involve only finite groups:

Definition 3. *Let H be a finite group acting on a finite set X. Then $F(X \times H)$ is called a transformation group algebra if it is equipped with the multiplication $\bullet$ given by*

$$
F_1 \bullet F_2(x,y) = \int_H F_1(x,z) F_2(z^{-1}x, z^{-1}y) dz\,. \tag{4.6}
$$

For a more general definition and references, see [31]. When we take $X = H$ and the action of H given by conjugation, then we regain the algebra structure of quantum double $D(H)$. There is a general theorem which classifies the irreducible representations of all transformation group algebras as defined above, but before we give this, we must first define the Hilbert spaces that the representations will act upon. Let N be a subgroup of H, let α be a unitary representation of N, and let V_α be its module, then we define

$$
F_\alpha(H,V_\alpha) := \left\{ \phi : H \to V_\alpha | \phi(xn) = \pi_\alpha(n^{-1})\phi(x), \forall x \in H, \forall n \in N \right\}. \tag{4.7}
$$

The irreps of our transformation group algebras are then described by the following theorem, which is a simple consequence of theorem 3.9 in [32]

Theorem 1. *Let $F(X \times H)$ be a transformation group algebra and let $\{\mathcal{O}_A\}$ be the collection of H-orbits in X (A takes values in some index set). For each A, choose some*

$\xi_A \in \mathcal{O}_A$ and let N_A be the stabilizer of ξ_A in H. Then, for each pair $(\mathcal{O}_A, \alpha)$ of an orbit $\mathcal{O}_A$ and an irrep α of the stabilizer N_A of this orbit, we have an irreducible unitary representation τ_α^A of $F(X \times H)$ on $F_\alpha(H, V_\alpha)$ given by

$$(\tau_\alpha^A(F)\phi)(x) := \int_H F(x\xi_A, z)\phi(z^{-1}x)dz\,. \tag{4.8}$$

Moreover, all unitary irreducible representations of $F(X \times H)$ are of this form and irreps τ_α^A and τ_β^B are equivalent only if $\mathcal{O}_A = \mathcal{O}_B$ and $\alpha \cong \beta$.

In the case of $D(H)$, the orbits $\mathcal{O}_A$ are just the conjugacy classes of H. The irreps Π_α^A of $D(H)$ are thus labeled by pairs (A, α) of a conjugacy class label A and an irrep α of the centralizer N_A of a specified element $g_A \in C_A$. We see that the spectrum of irreps of $D(H)$ is in one to one correspondence with the spectrum of excitations of the discrete gauge theory that we described in section 2. In particular, the pure (uncharged) magnetic fluxes correspond to the Π_1^A, where 1 denotes the trivial representation of H, and the pure charges correspond to the Π_α^e. We will call the element $g_A \in A$ the *preferred element* of A. Any choice of preferred element yields the same isomorphy class of representations of $D(H)$. The carrier space of Π_α^A is just the space $F_\alpha(H, V_\alpha)$ defined above and for brevity we will denote it by V_α^A The dimension d_α^A of V_α^A is the product of the number of elements of the conjugacy class A and the dimension d_α of α. To see this, note that the functions in V_α^A are completely determined once their value on one element of each N_A-coset is chosen. Now the number of N_A-cosets is just $|A|$, the number of elements of A, which shows that $d_\alpha^A = |A| d_\alpha$. In fact, there is a canonical correspondence between cosets of N_A and elements of A: the coset hN_A corresponds to the element hg_Ah^{-1}. Thus, a state with pure flux hg_Ah^{-1} will be represented by a wave function with support on hN_A.

The action of an element $F \in D(H)$ on V_α^A is given by the formula in the theorem above, which in the case of $D(H)$ becomes

$$\left(\pi_\alpha^A(F)\phi\right)(x) := \int_H dz\, F(xg_Ax^{-1}, z)\, \phi(z^{-1}x)\,. \tag{4.9}$$

From this formula, it is easy to see that the action of the gauge group elements $1 \otimes g$ in the purely electric representation Π_α^e is indeed isomorphic to the action of the gauge group in the representation α. Also, the action of the gauge group on magnetic fluxes is given by conjugation, which can be seen as follows: the state with flux hg_Ah^{-1} is represented by the function $\delta_{hN_A} \in V_1^A$, i.e. by the characteristic function of the coset hN_A. The action of the element $1 \otimes g \in D(H)$ sends this function to the function ghN_A, which in turn corresponds to the flux $g(hg_Ah^{-1})g^{-1}$.

The *spin* of a particle that transforms in the irrep Π_α^A is given by the action of the ribbon element c. We have

$$\left(\pi_\alpha^A(c)\phi\right)(x) := \alpha(g_A^{-1})\phi(x)\,. \tag{4.10}$$

Since the element g_A^{-1} is central in N_A, the matrix $\alpha(g_A^{-1})$ is a constant multiple of the unit matrix: we have $\alpha(g_A^{-1}) = s_\alpha^A I$, where $s_\alpha^A \in \mathbb{C}$ is a root of unity which we call the *spin factor* of Π^A_α. Clearly, we have $s^A_\alpha = \frac{1}{\mathrm{d}_\alpha}\mathrm{Tr}(\alpha(\mathrm{g}_\mathrm{A}^{-1}))$, where d_α is the dimension of the representation α of N_A. A consistent description of the *braiding* for arbitrary representations of $D(H)$ is given by the R-matrix (4.2).

4.3 Matrix elements and characters

The fusion rules for representations of $D(H)$ and of more general transformation group algebras may be calculated by means of a character formalism. The character of a representation π of a transformation group algebra $\mathcal{T} = C_c(X \times H)$ is a linear functional $\chi : \mathcal{T} \to \mathbb{C}$ defined as follows: $\chi(t)$ is the trace of the matrix $\pi(t)$. Clearly, the character of the representation π depends only on the isomorphism class of π. Let us calculate the character of an irrep τ^A_α of $\mathcal{T}$ from the formula in theorem 1. The first thing we need is a basis for the vector space $F_\alpha(G, V_\alpha)$. To get this, we choose a basis e^α_i for V_α and a representative for each left coset of N_A. For a given coset of N_A, all the elements of this coset send the preferred element ξ_A of the orbit $\mathcal{O}_A$ to the same element ζ of this orbit. Moreover, each element $\zeta \in \mathcal{O}_A$ uniquely determines a coset. Therefore, we call the representative elements of the cosets x_ζ and these x_ζ are just arbitrarily chosen elements of H with the property that $x_\zeta \xi_A = \zeta$. A basis for $F_\alpha(G, V_\alpha)$ is now given by the functions ϕ^i_ζ defined by

$$\phi^i_\zeta(y) = 1_{x_\zeta N_A}(y)\alpha(y^{-1}x_\zeta)e^\alpha_i\,. \tag{4.11}$$

ϕ^i_ζ is the unique element of $F_\alpha(G, V_\alpha)$ which takes the value e^α_i at x_ζ and one may easily verify that these elements do indeed form a basis for $F_\alpha(G, V_\alpha)$. The matrix elements of τ^A_α in this basis are given by

$$\tau^A_\alpha(F)^{i,j}_{\zeta,\eta} = \int_{N_A} F\left(x_\eta\xi_A, x_\eta n x_\zeta^{-1}\right)\alpha_{i,j}(n)\,dn\,, \tag{4.12}$$

where the $\alpha_{i,j}$ are the matrix elements of α with respect to the basis of e^α_i. As a consequence, the character χ^A_α of τ^A_α is given by

$$\chi^A_\alpha(F) = \int_{\mathcal{O}_A} d\zeta \int_{N_A} dn\, F\left(x_\zeta\xi_A, x_\zeta n x_\zeta^{-1}\right)\chi_\alpha(n)\,, \tag{4.13}$$

where χ_α denotes the character of α. We can remove the arbitrarily chosen elements x_ζ from this formula by adding an integration over N_A, thus changing the integration over $\mathcal{O}_A$ into an integration over H:

$$\chi^A_\alpha(F) = \int_H dz \int_{N_A} dn\, F(z\xi_A, znz^{-1})\chi_\alpha(n)\,. \tag{4.14}$$

When $\mathcal{T} = D(H)$, this reduces to the formula given in [30]. Clearly, the characters are fully determined by their values on a basis for $\mathcal{T}$. When X is finite, we can take the basis of delta functions $\delta_\eta\delta_h$ ($\eta \in X, h \in H$) and we can take the characters to be elements of $F(X \times H)$ by writing $\chi^A_\alpha(\eta, h) := \chi^A_\alpha(\delta_\eta\delta_h)$. We have

$$\chi^A_\alpha(\eta, h) = 1_{N_\eta}(h)1_{\mathcal{O}_A}(\eta)\chi_\alpha(x_\eta^{-1}hx_\eta)\,. \tag{4.15}$$

When $\mathcal{T} = D(H)$, this gives the formula for the characters in [28].

We may define an inner product $\langle \cdot, \cdot \rangle$ on the space of functions $X \times H$ by the formula

$$\langle \chi_1, \chi_2 \rangle = \int_X d\eta \int_H dh \, \chi_1(\eta, h) \overline{\chi_2(\eta, h)} \, . \tag{4.16}$$

One may check that the characters are orthogonal with respect to this inner product:

$$\left\langle \chi_\alpha^A, \chi_\beta^B \right\rangle = |H| \delta_{A,B} \delta_{\alpha,\beta} \, . \tag{4.17}$$

When $\mathcal{T} = D(H)$, the inner product defined here is just the invariant inner product (4.5) on $D(H)^*$ and the orthogonality of the characters with respect to this inner product follows from Woronowicz's general theory. The decomposition of a tensor product of irreps of $D(H)$ may be found by calculating the inner products of the characters of the irreps with the character of the tensor product. In this way, the fusion properties of pure fluxes and charges that we have described in section 2 are reproduced and we may also calculate the fusion rules for dyons.

5. Examples of quantum doubles

We briefly present examples of quantum doubles of finite groups. These will be the standard examples in our discussion of symmetry breaking in the remainder of the paper.

5.1 $D(H)$ for abelian H

The quantum double of an abelian group H is isomorphic to the group algebra of $H \times H$ as a Hopf algebra. One way to see this is the following: First recall that any finite abelian group is isomorphic to some $\mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_n}$, where $k_i | k_j$ for $i < j$. Then recall that, by Pontryagin duality, the character group of an abelian group H is isomorphic to H. We may thus denote the elements of H by n-tuples $(m_1, \ldots, m_n)$, with $0 \leq m_i \leq k_i$ and we may also label the characters $\chi_{m_1,\ldots,m_n}$ of H with such n-tuples in such a way that the map $(m_1, \ldots, m_n) \to \chi_{m_1,\ldots,m_n}$ is an isomorphism of groups. The canonical way to do this labeling is such that $\chi_{m_1,\ldots,m_n}$ is the character given by

$$\chi_{m_1,\ldots,m_n}(l_1, \ldots, l_n) = \exp\left(2\pi i \left(\frac{m_1 l_1}{k_1} + \cdots + \frac{m_n l_n}{k_n}\right)\right) . \tag{5.1}$$

The characters are linearly independent functions on H and thus $D(H)$ is spanned by the elements $\chi \otimes \delta_h$, where χ is a character of H and h is an element of H. But one calculates easily that

$$\begin{aligned}
(\chi_1 \otimes \delta_{h_1}) \bullet (\chi_2 \otimes \delta_{h_2}) &= (\chi_1 \chi_2 \otimes \delta_{h_1 h_2}) \\
1_{D(H)} &= 1 \otimes \delta_e \equiv e_{H \times H} \\
\Delta(\chi \otimes \delta_h) &= (\chi \otimes \delta_h) \otimes (\chi \otimes \delta_h) \\
S(\chi \otimes \delta_h) &= (\bar{\chi} \otimes \delta_{h^{-1}}) = (\chi \otimes \delta_h)^{-1} \\
\epsilon(\chi \otimes \delta_h) &= 1 \, ,
\end{aligned} \tag{5.2}$$

so that, comparing to (3.8), we see that we indeed have $D(H) \cong \mathbb{C}(H \times H)$ as a Hopf algebra.

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As a consequence, the irreducible representations of $D(H)$ are just the tensor products $\chi_1 \otimes \chi_2$ that may be formed from two irreps χ_1, χ_2 of H. These correspond to the irreps Π^A_α that we described in section 4.2 in the following way. When H is abelian, all conjugacy classes of A consist of just one element, so that the label A may be identified with the element g_A of H. This element may in turn be identified with a character χ_A using the isomorphism between H and its character group that we indicated above. Also, we have $N_A = H$ and so α is already a character of H. One may now check easily that the irrep Π^A_α of $D(H)$ corresponds to the irrep $\chi_A \otimes \alpha$ of $H \otimes H$. The tensor product of two $D(H)$-irreps is just the usual tensor product of $H \times H$ representations. The only difference with the usual representation theory of $H \times H$ is that the representations now have non-trivial spin-factors and non-trivial braiding, given by the ribbon element and the R-matrix of $D(H)$ respectively. We have

$$\begin{aligned} s^A_\alpha &= \alpha(g_A^{-1}) \\ \Pi^A_\alpha \otimes \Pi^B_\beta(R) &= \beta(g_A) \,. \end{aligned} \tag{5.3}$$

As one may read off using (5.1), these are just the usual phase factors one expects for abelian dyons, involving products of charge and flux quantum numbers.

5.2 $D(D_{2m+1})$

Perhaps the simplest non-abelian groups are the dihedral groups D_n which describe the symmetries of the regular n-gons. Among these is the smallest non-abelian group: the dihedral group D_3, which is isomorphic to the symmetric group S_3. D_n has $2n$ elements: the unit, $n-1$ non-trivial rotations and n reflections. It can be presented on two generators as follows:

$$D_n = \{s, r |\, s^2 = r^n = 1, sr = r^{n-1}s\} \,. \tag{5.4}$$

The $2n$ elements may all be written in the form r^k or sr^k (with $e = r^0$). The powers of r are the rotations, the elements that involve s are the reflections.

In this paper, we will deal exclusively with the odd dihedral groups D_{2m+1}. D_{2m+1} has $m+2$ conjugacy classes, which we will label by their preferred elements and which we will denote by their preferred elements in square brackets (e.g.$[r]$) if confusion between class and element might arise. The classes are

$$\begin{aligned} [e] &= \{e\} \\ [r^k] &= \{r^k, r^{-k}\} \quad (0 < k \leq m) \\ [s] &= \{sr^k |\, 0 \leq k \leq 2m+1\} \,. \end{aligned} \tag{5.5}$$

The centralizers of these classes are given by

$$N_e = D_{2m+1} \,, \qquad N_{r^k} = \langle\, r \,\rangle \cong \mathbb{Z}_{2m+1} \,, \qquad N_s = \langle\, s \,\rangle \cong \mathbb{Z}_2 \,. \tag{5.6}$$

D_{2m+1} has two one dimensional representations: the trivial representation, which we will denote J_0 and a representation J_1 given by $J_1(r) = 1,\ J_1(s) = -1$. The remaining irreps

	$[e]$	$[r^k]$	$[s]$
J_0	1	1	1
J_1	1	1	-1
α_j	2	$q^{jk}+q^{-jk}$	0

Table 1: Character table for D_{2m+1}. We have defined $q = e^{2\pi i/(2m+1)}$.

of D_{2m+1} are all two dimensional and faithful. We will call them $\alpha_1, \ldots, \alpha_m$ and they may be given by

$$\alpha_k(r) = \begin{pmatrix} \cos(\frac{2k\pi}{2m+1}) & -\sin(\frac{2k\pi}{2m+1}) \\ \sin(\frac{2k\pi}{2m+1}) & \cos(\frac{2k\pi}{2m+1}) \end{pmatrix} \qquad \alpha(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.7}$$

The character table for D_{2m+1} can now be read off; it is given in table 1. The fusion rules of the irreps of $D(D_{2m+1})$ may be determined by means of the characters (4.15) and the orthogonality relations (4.17). They have been given explicitly in [33]. One may also show that tensor products of multiple $D(D_{2m+1})$-irreps can carry non-abelian representations of the braid group.

6. Symmetry breaking

6.1 Hopf symmetry breaking

Consider the situation where a condensate has formed, carrying the representation Π^A_α of $D(H)$. The ground state or "vacuum" of the theory is then a background of identical particles, all in the same state $\phi \in V^A_\alpha$. We model this situation with a tensor product state $\phi \otimes \phi \otimes \cdots \otimes \phi$. This state breaks the $D(H)$-symmetry of the theory and we want to find out what the residual symmetry algebra of the system after this breaking is. Now if the original symmetry were described by a group, then finding the residual symmetry would in principle be straightforward; we would find out which of the group elements leave the condensate state ϕ invariant, i.e. we would find the stabilizer of ϕ, and this stabilizer would be the residual symmetry. If the original symmetry is described by a Hopf algebra, then we cannot use this recipe, for several reasons. First of all, we cannot expect to find a subalgebra of the Hopf algebra which leaves ϕ invariant in the usual sense of the word; any such subalgebra would have to contain the element 0 which would obviously send ϕ to 0. Hence, we need a new definition of invariance. Fortunately, there is a natural definition, namely the following (cf. [17])

Definition 4. *Let $\mathcal{A}$ be a Hopf algebra with counit ϵ, let $a \in \mathcal{A}$ and let ϕ be a vector in some $\mathcal{A}$-module. Then we say that ϕ is left invariant by a if the action of a on ϕ is given by $a \cdot \phi = \epsilon(a)\phi$.*

This definition is natural, since it just says that the vector ϕ transforms under $a \in \mathcal{A}$ in the same way that the vacuum would. Also, if the Hopf algebra $\mathcal{A}$ is a group algebra, then we see that this definition of invariance reduces to the usual one on the group elements.

Nevertheless, when we apply the above definition of invariance to a group algebra, then we see that the subalgebra which leaves a vector ϕ invariant is *not* the group algebra of the stabilizer of ϕ. In fact, it is a much larger algebra, which is not a Hopf algebra. On the other hand, the maximal Hopf subalgebra of the group algebra which leaves ϕ invariant (with the above definition of invariance), is exactly the group algebra of the stabilizer of ϕ. This follows easily from the fact that the Hopf subalgebras of a group algebra $\mathbb{C}G$ are exactly the group algebras of the subgroups of G (this is well known to Hopf algebra theorists, but we also give an explanation of why it is so in section 6.3). This suggests that we should define the residual symmetry algebra after breaking as follows:

Definition 5. *Suppose we have a theory with Hopf symmetry $\mathcal{A}$. If a condensate of particles in the state ϕ forms in this theory, then the residual symmetry algebra is the maximal Hopf subalgebra of $\mathcal{A}$ that leaves ϕ invariant. We will call this algebra the* Hopf stabilizer *of ϕ*

A maximal Hopf subalgebra with a certain property is defined as a Hopf subalgebra with this property which is not a subalgebra of a larger Hopf subalgebra with his property. The maximal Hopf subalgebra in the above definition is unique, since, if we have two different Hopf subalgebras which leave the same vector invariant, then the subalgebra generated by these two is itself a Hopf subalgebra which leaves this vector invariant and which contains the original two Hopf subalgebras. The above definition reduces to the usual definition in the case of group algebras and it has the further advantage that the residual symmetry algebra will always be a Hopf algebra.[3] The spectrum of the residual algebra will thus have the desirable properties that we discussed in section 3; there will be a natural description of many-particle states, there will be a trivial or vacuum representation and given an irrep of the algebra that labels a "particle" (an excitation over the condensate), there will also be an irrep (possibly the same) that labels the "antiparticle". The fact that the residual symmetry algebra is a Hopf algebra also makes sure that the invariance of ϕ implies the invariance of all the states $\phi\otimes\phi\otimes\ldots\otimes\phi$. This follows easily from the fact that $(\epsilon\otimes\epsilon)\circ\Delta = \epsilon$. Thus, we might have taken the condensate to be a superposition of states with different numbers of particles (still all in the state ϕ) and this would have yielded exactly the same residual algebra.

6.2 Hopf subalgebras and Hopf quotients

In view of the definition 5 of the residual symmetry algebra after the formation of a condensate, it is useful to find out all we can about Hopf subalgebras of the quantum double $D(H)$, or more generally, about Hopf subalgebras of finite dimensional semisimple Hopf algebras. In the present section, we give a characterization of the Hopf subalgebras of such Hopf algebras, which will provide us with a way of finding all these Hopf subalgebras and in particular the residual symmetry algebras of definition 5 in a systematic way. Along with the results on Hopf subalgebras, we also prove some results on Hopf quotients or quotient

[3] In fact, it is also semisimple since any Hopf subalgebra of a finite dimensional semisimple Hopf algebra is itself semisimple. This is proved in [14], using the Nichols-Zoeller theorem [34].

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Hopf algebras which will be useful in our discussion of confinement further on. The main theorems in this section are closely related to results in [35] and [36]. We include elementary proofs here in order to make the paper more self-contained. We write the results in a form which is useful for our needs, rather than maximally general or compact. A lot of background for this section can be found in [14].

We define a Hopf quotient as follows

Definition 6. *Let $\mathcal{A}$ and $\mathcal{B}$ be Hopf algebras. If we have a surjective Hopf map $\Gamma : \mathcal{A} \to \mathcal{B}$, then we call $\mathcal{B}$ a* Hopf quotient *of $\mathcal{A}$.*

The Hopf algebra $\mathcal{B}$ is in fact isomorphic to the quotient of $\mathcal{A}$ by the kernel of the map Γ, explaining the terminology. Our first step in characterizing Hopf subalgebras and Hopf quotients is to relate them to each other, using the following proposition

Proposition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite dimensional Hopf algebras and let $\Gamma : \mathcal{A} \to \mathcal{B}$ be a Hopf map. Then the dual map $\Gamma^* : \mathcal{B}^* \to \mathcal{A}^*$ is also a Hopf map. Moreover, if Γ is injective then Γ^* is surjective and if Γ is surjective then Γ^* is injective. Finally, if we identify $\mathcal{A}$ and $\mathcal{A}^{**}$ and $\mathcal{B}$ and $\mathcal{B}^{**}$ in the canonical way, then we have $\Gamma^{**} = \Gamma$.*

Proof. The proof that Γ^* is a Hopf map is straightforward calculation. As an example, we show that $\Gamma^* \otimes \Gamma^* \circ \Delta_{\mathcal{B}^*} = \Delta_{\mathcal{A}^*} \circ \Gamma^*$. For any $f \in \mathcal{B}^*$, we have

$$\Gamma^* \otimes \Gamma^* \circ \Delta_{\mathcal{B}^*}(f) = f \circ \mu_{\mathcal{B}} \circ \Gamma \otimes \Gamma = f \circ \Gamma \circ \mu_{\mathcal{A}} = \Delta_{\mathcal{A}^*} \circ \Gamma^*(f)\,. \tag{6.1}$$

We used the fact that Γ is a Hopf map in the second equality. The other properties that make Γ^* into a Hopf map can be verified analogously. The proof that $\Gamma^{**} = \Gamma$ is also easy. We identify the element $a \in \mathcal{A}$ with the functional $E_a \in \mathcal{A}^{**}$ that is evaluation in A, i.e. $E_a : f \to f(a)$. Similarly, we identify $b \in \mathcal{B}$ with $E_b \in \mathcal{B}^{**}$. One may check that these identifications are Hopf isomorphisms. The action of Γ^{**} on a is calculated as follows:

$$\Gamma^{**}(a)(f) = \Gamma^{**}(E_a)(f) = E_a \circ (\Gamma^*)(f) = E_a(f \circ \Gamma) = E_{\Gamma(a)}(f) = \Gamma(a)(f)\,. \tag{6.2}$$

Hence, we see that indeed $\Gamma^{**} = \Gamma$. The statements about injectivity or surjectivity of Γ^* are basic properties of the dual map. □

From this proposition, we have the following corollaries

Corollary 1. *Let $\mathcal{A}$ be a finite dimensional Hopf algebra and let $\mathcal{B}$ be a Hopf subalgebra of $\mathcal{A}$. Then $\mathcal{B}^*$ is a Hopf quotient of $\mathcal{A}^*$. The corresponding surjective Hopf map is restriction to $\mathcal{B}$, which is the dual map of the embedding of $\mathcal{B}$ in $\mathcal{A}$*

Corollary 2. *Let $\mathcal{A}$ be a finite dimensional Hopf algebra and let $\mathcal{B}$ be a Hopf quotient of $\mathcal{A}$, with corresponding surjective Hopf map Γ. Then $\mathcal{A}^*$ has a Hopf subalgebra isomorphic to $\mathcal{B}^*$, namely the image of the embedding Γ^*.*

The next proposition shows that if $\mathcal{B}$ is a Hopf quotient of $\mathcal{A}$, then the set of representations of $\mathcal{B}$ naturally corresponds to a subring of the representation ring of $\mathcal{A}$.

Proposition 2. *Let $\mathcal{A}$ be a Hopf algebra and let $\mathcal{B}$ be a Hopf quotient of $\mathcal{A}$. Denote the associated surjective Hopf map from $\mathcal{A}$ to $\mathcal{B}$ by Γ. Then the representations of $\mathcal{B}$ are in one-to-one correspondence with the representations of $\mathcal{A}$ that factor over Γ. Also, this correspondence preserves irreducibility. It follows that, if $\mathcal{A}$ is semisimple, then so is $\mathcal{B}$. The correspondence map between representations also commutes with taking conjugates and tensor products of representations. As a consequence, the tensor product of irreps of $\mathcal{A}$ that factor over Γ will decompose in the same way as the tensor product of the corresponding irreps of $\mathcal{B}$.*

Proof. Let ρ be a representation of $\mathcal{B}$. Then $\rho \circ \Gamma$ is a representation of $\mathcal{A}$, since Γ is a Hopf map. Moreover, if ρ is irreducible then so is $\rho \circ \Gamma$, since Γ is surjective. On the other hand, let $\tau : \mathcal{A} \to M_{n\times n}$ be a representation of $\mathcal{A}$ which factors over Γ, that is, $\tau = \rho \circ \Gamma$ for some map $\rho : \mathcal{B} \to M_{n\times n}$. Then ρ is a representation of $\mathcal{B}$, since Γ is surjective and irreducibility of τ implies that ρ is irreducible. Also, τ uniquely determines ρ and vice versa. Hence, the representations of $\mathcal{B}$ are in one-to-one correspondence with the representations of $\mathcal{A}$ which factor over Γ and irreducibility is preserved in this correspondence. Semisimplicity of $\mathcal{A}$ is equivalent to the property that all $\mathcal{A}$-modules decompose into irreducibles. This holds in particular for all $\mathcal{A}$-modules in which the action of $\mathcal{A}$ factors over Γ, and hence also for all $\mathcal{B}$-modules, implying that $\mathcal{B}$ is semisimple. The remaining statements follow easily from the fact that Γ is a Hopf map. If $\tau = \rho \circ \Gamma$ then $\overline{\tau} = \overline{\rho} \circ \Gamma$. This can be seen by looking at the matrix elements $\overline{\tau}_{i,j}$ of $\overline{\tau}$:

$$\overline{\tau}_{i,j} = (\tau_{j,i} \circ S_{\mathcal{A}}) = \rho_{j,i} \circ \Gamma \circ S_{\mathcal{A}} = \rho_{j,i} \circ S_{\mathcal{B}} \circ \Gamma = \overline{\rho}_{i,j} \otimes \Gamma\,. \tag{6.3}$$

Here, we have used $\Gamma \circ S_{\mathcal{A}} = S_{\mathcal{B}} \circ \Gamma$. For the tensor product of irreps $\tau^a = \rho \circ \Gamma$ and $\tau^b = \rho^b \circ \Gamma$, we have

$$\tau^a \otimes \tau^b \circ \Delta_{\mathcal{A}} = \rho^a \otimes \rho^b \circ \Gamma \otimes \Gamma \circ \Delta_{\mathcal{A}} = \rho^a \otimes \rho^b \circ \Delta_{\mathcal{B}} \circ \Gamma\,, \tag{6.4}$$

where we used that $\Gamma \otimes \Gamma \circ \Delta_{\mathcal{A}} = \Delta_{\mathcal{B}} \circ \Gamma$. We see that $\tau^a \otimes \tau^b \circ \Delta_{\mathcal{A}}$ and $\rho^a \otimes \rho^b \circ \Delta_{\mathcal{B}}$ act on the same module by the same matrices (since Γ is surjective). Hence the decomposition of tensor product representations of $\mathcal{A}$ will be the same as the decomposition of tensor product representations of $\mathcal{B}$. $\square$

Before the next proposition, we need another definition

Definition 7. *We call a set X of irreps of a Hopf algebra $\mathcal{A}$* closed under tensor products and conjugation *if $\tau \in X \Rightarrow \overline{\tau} \in X$ and if $\tau^a, \tau^b \in X$ implies that all the irreps in the decomposition of the tensor product of τ^a and τ^b are contained in X.*

Note that, in the previous proposition, the set of irreps of $\mathcal{A}$ that factor over Γ is an example of a set of irreps of $\mathcal{A}$ that close under tensor products and conjugation. Also note that a set of irreps that closes under tensor products and conjugation will always contain the counit. We now prove a basic fact about sets of irreducibles of $\mathcal{A}$ that close under conjugation and tensor products:

Proposition 3. *Let $\mathcal{A}$ be a semisimple Hopf algebra and let X be a set of irreps of $\mathcal{A}$ that closes under conjugation and tensor products. Then the linear space V_X spanned by the matrix elements of the representations in X is a Hopf subalgebra of $\mathcal{A}^*$*

Proof. First, let us take the product of two matrix elements. We have $\mu_{\mathcal{A}^*}(\tau^a_{i,j}\tau^b_{k,l}) = (\tau^a \otimes \tau^b \circ \Delta_{\mathcal{A}})_{(i,j),(k,l)}$. In other words, the product of matrix elements of τ^a and τ^b in $\mathcal{A}^*$ is a matrix element of the tensor product representation $\tau^a \otimes \tau^b \circ \Delta_{\mathcal{A}}$. Since $\mathcal{A}$ is semisimple, this tensor product may be decomposed into irreps and the matrix elements of the tensor product representation are linear combinations of the matrix elements of the irreps this decomposition. But since these irreps are contained in X, it follows that V_X is closed under multiplication. Clearly, V_X also contains $1_{\mathcal{A}^*} = \epsilon_A$, so V_X is a unital subalgebra of $\mathcal{A}^*$. We also have $S(V_X) \subset V_X$, since

$$S_{\mathcal{A}^*}(\tau_{i,j}) = \tau_{i,j} \circ S_{\mathcal{A}} = \overline{\tau}_{j,i} \tag{6.5}$$

and $\tau \in X \Rightarrow \overline{\tau} \in X$. For the comultiplication of a matrix element of any representation of $\mathcal{A}$, we have

$$\Delta_{\mathcal{A}^*}(\tau_{i,j}) = \sum_k \tau_{i,k} \otimes \tau_{k,j} \tag{6.6}$$

and hence we have $\Delta_{\mathcal{A}^*}(V_X) \subset V_X \otimes V_X$. □

Now we arrive at one of our main goals, which is a partial converse of the previous proposition:

Theorem 2. *Let $\mathcal{A}$ be a finite dimensional semisimple Hopf algebra over the complex numbers. Let $\mathcal{B}$ be a Hopf subalgebra of $\mathcal{A} = \mathcal{A}^{**}$. Then $\mathcal{B}$ is spanned by the matrix elements of a set of irreps of $\mathcal{A}^*$ which closes under conjugation and tensor products.*

Proof. Let ι be the inclusion of $\mathcal{B}$ into $\mathcal{A}$. Then $\mathcal{B}^*$ is a Hopf quotient of $\mathcal{A}^*$ with the associated Hopf map given by ι^* (cf. corollary 1). Because $\mathcal{A}$ is semisimple and defined over the complex numbers, its dual $\mathcal{A}^*$ is also semisimple (see [37] and also [14]). Hence, using proposition 2, $\mathcal{B}^*$ is also semisimple. But then it follows that the matrix elements of the irreps of $\mathcal{B}^*$ span $\mathcal{B}^{**} = \mathcal{B}$. On the other hand, we know from proposition 2 that the irreps of $\mathcal{B}^*$ are identified (through $\iota^{**} = \iota$) with a set of irreps of $\mathcal{A}^*$ which closes under conjugation and tensor products. □

This theorem is the characterization of Hopf subalgebras that we will be using in our discussion of $D(H)$-symmetry breaking in section 6.3. More generally, it can simplify the problem of finding all the Hopf subalgebras of a finite dimensional semisimple Hopf algebra $\mathcal{A}$ enormously. If the irreps of $\mathcal{A}^*$ and the decompositions of their tensor products are known, then finding all sets of irreps of $\mathcal{A}^*$ that close under tensor products is a process that can be carried out easily on a computer. Finally, we prove a similar statement about Hopf quotients:

Theorem 3. *Let $\mathcal{A}$ be a finite dimensional semisimple Hopf algebra over the complex numbers. Then any Hopf quotient of $\mathcal{A} = \mathcal{A}^{**}$ is isomorphic to a quotient obtained by restriction to a Hopf subalgebra of $\mathcal{A}^*$ generated by matrix elements of a set of irreps of $\mathcal{A}$ which closes under conjugation and tensor products.*

Proof. Let $\mathcal{B}$ be a Hopf quotient of $\mathcal{A}$ and let Γ be the associated projection. Then $\Gamma^*(\mathcal{B}^*) \cong \mathcal{B}^*$ is a Hopf subalgebra of $\mathcal{A}^*$ (cf. corollary 2) and $\mathcal{B} = \mathcal{B}^{**}$ is isomorphic to the quotient of $\mathcal{A} = \mathcal{A}^{**}$ obtained by restriction to $\Gamma^*(\mathcal{B}^*)$. Since $\mathcal{A}$ is semisimple, $\mathcal{A}^*$ is also semisimple. Thus, we can now apply the previous theorem to the pair $(\mathcal{A}^*, \Gamma^*(\mathcal{B}^*))$ and it follows that $\Gamma^*(\mathcal{B}^*)$ is the desired Hopf subalgebra. □

6.3 Hopf subalgebras of quantum doubles

In section 6.2, we showed that the Hopf subalgebras of a finite dimensional semisimple Hopf algebra (such as $D(H)$) are in one-to-one correspondence with sets of irreps of the dual Hopf algebra that close under tensor products and conjugation. Therefore, we now construct the representations of the dual algebra $D(H)^*$. From (4.4), we see that, as an algebra (but not as a Hopf algebra), $D(H)^*$ is isomorphic to $\mathbb{C}H \otimes F(H)$. As a consequence, the irreducible representations of $D(H)^*$ are tensor products of irreps of $\mathbb{C}H$ and irreps of $F(H)$. The irreps of $\mathbb{C}H$ just correspond to the irreps of H and we will denote them ρ_i. The irreps of $F(H)$ are all one dimensional and are labeled by the elements of H. We have an irrep E_g for each $g \in H$, given by

$$E_g(f) = f(g)\,. \tag{6.7}$$

We can thus label each representation of $D(H)^*$ by a pair (ρ_i, g). Tensor products of the irreps of $D(H)^*$ may be formed by means of Δ^*. Although this coproduct is not the same as the usual coproduct for $\mathbb{C}H \otimes F(H)$, the decomposition of tensor products into irreps is not affected by this (the Clebsch-Gordan coefficients for the decomposition are affected). Thus we have

$$\rho_i \otimes \rho_j = \bigoplus_k N_{ij}^k \rho_k \implies (\rho_i, g) \otimes (\rho_j, h) = \bigoplus_k N_{ij}^k (\rho_k, gh) \tag{6.8}$$

where the N_k^{ij} are the usual multiplicities in the decomposition of tensor products of H-irreps. Also, we have

$$\overline{(\rho_i, g)} = (\bar{\rho}_i, g^{-1})\,. \tag{6.9}$$

From these formulae, we see that any set X of irreps of $D(H)^*$ that closes under tensor products and conjugation is associated to a set of irreps of H and a set of irreps of $F(H)$ with the same property. These sets just consist of the irreps that may occur as a factor of one of the irreps in X. Consequently, for any Hopf subalgebra $\mathcal{B}$ of $D(H)$, there are minimal Hopf subalgebras $\mathcal{C}$ of $F(H)$ and $\mathcal{D}$ of $\mathbb{C}H$ such that $\mathcal{B} \subset \mathcal{C} \otimes \mathcal{D} \subset D(H)$. In the other direction, we see that, for any pair of Hopf subalgebras $\mathcal{C} \subset F(H)$ and $\mathcal{D} \subset \mathbb{C}H$, the vector space $\mathcal{C} \otimes \mathcal{D}$ is a Hopf subalgebra of $D(H)$. Note that this Hopf subalgebra is usually not isomorphic to $\mathcal{C} \otimes \mathcal{D}$ as a Hopf algebra. Also, not all Hopf subalgebras of $D(H)$ are of this form. However, the ones that are will be quite important in the sequel. Therefore, we now find all the Hopf subalgebras of the group algebra $\mathbb{C}H$ and of the function algebra $F(H)$. This also gives us two simple examples of the use of theorem 2.

Proposition 4. *The Hopf subalgebras of a group algebra $\mathbb{C}H$ are the group algebras of the subgroups of H.*

Proof. The Hopf algebra dual to $\mathbb{C}H$ is the vector space $F(H)$ of functions on H, with Hopf algebra structure given by

$$\begin{aligned} 1^* : g \mapsto 1\,, \qquad \mu^*(f_1, f_2) : g \mapsto f_1(g) f_2(g)\,, \qquad \Delta^*(f) : (g_1, g_2) \mapsto f(g_1 g_2)\,, \\ \epsilon^* : f \mapsto f(e)\,, \qquad S^*(f) : g \mapsto f(g^{-1})\,, \end{aligned} \tag{6.10}$$

where g, g_1, g_2 are arbitrary elements of the group H. Note that, in the formula for the comultiplication, we have identified $F(H) \otimes F(H)$ with $F(H \times H)$ in the usual way. The irreducible representations E_g of $F(H)$ were given in (6.7). One checks easily that the tensor product of two if these irreps, as defined using Δ^*, is given by

$$E_g \otimes E_h = E_{gh}\,. \tag{6.11}$$

Also, we have $\bar{\pi}_g = E_{g^{-1}}$. Hence, the sets of irreps of $(\mathbb{C}H)^*$ that close under conjugation and tensor products correspond exactly to the subgroups of H. The proposition follows. □

Proposition 5. *A Hopf subalgebra of an algebra $F(H)$ of functions on a group H is isomorphic to the algebra $F(H/K)$ of functions on the quotient of H by some normal subgroup K.*

Proof. Let $\mathcal{C}$ be a Hopf subalgebra of $F(H)$ and let us denote the irreps of $\mathbb{C}H$ whose matrix elements span $\mathcal{C}$ by ρ_i. Then the intersection of the kernels of the ρ_i is a normal subgroup K of H and any function in $\mathcal{C}$ will be constant on the cosets of K. We can also say that $\mathcal{C}$ really consists of functions on the quotient group H/K. Now let us show the opposite inclusion. If we form the direct sum $\oplus_i \rho_i$ of all the representations ρ_i, then this representation of H will have exactly K as its kernel and hence it can be identified with a faithful representation of H/K. Now it is a theorem in the theory of finite groups that the tensor powers of any faithful representation of a group contain all irreducible representations of this group (see for instance [38]). Hence all irreps of H/K are contained in the tensor powers of $\oplus_i \rho_i$ and hence the matrix elements of these irreps are contained in $\mathcal{C}$. But since the matrix elements of the irreps of H/K span $F(H/K)$, it follows that $\mathcal{C} \cong F(H/K)$. □

Thus we see that for any Hopf subalgebra $\mathcal{B}$ of $D(H)$, there is a maximal normal subgroup K of H and a minimal subgroup N of H such that $\mathcal{B}$ is in fact a Hopf subalgebra of $F(H/K) \otimes \mathbb{C}N$. Also, every subalgebra of $D(H)$ of the form $F(H/K) \otimes \mathbb{C}N$ is automatically a Hopf subalgebra. These particular Hopf subalgebras are in fact also transformation group algebras, with the group N acting on H/K by conjugation. This will be very useful later on, since it will allow us to apply the representation theory of transformation group algebras that we described in section 4.

Now let us turn to the problem of finding the Hopf subalgebra of $D(H)$ which leaves a given condensate vector ϕ invariant.

Proposition 6. *The Hopf stabiliser $\mathcal{T} \subset D(H)$ of a given vector $\phi \in V_\alpha^A$ is spanned by the matrix elements of those irreps (ρ, g) of $D(H)^*$ for which*

$$\forall x : \phi(gx) = \frac{\chi_\rho(g_A)}{d_\rho} \phi(x)\,. \tag{6.12}$$

Here, χ_ρ denotes the character of the irrep ρ of H and d_ρ denotes its dimension. This equation for (ρ, g) can only be satisfied if $\frac{\chi_\rho(g_A)}{d_\rho}$ is a root of unity. $\mathcal{T}$ is a transformation group algebra of the form $F(H/K) \otimes \mathbb{C}N$ if and only if this root of unity equals 1 for all (ρ, g) which satisfy (6.12)

Proof. $\mathcal{T}$ is by definition the maximal Hopf subalgebra of $D(H)$ which leaves ϕ invariant. Therefore it is spanned by the matrix elements of a set of irreps (ρ_i, g) of $D(H)^*$ which closes under conjugation and tensor products (cf. theorem 2). The requirement that the matrix elements of the irrep (ρ_i, g) leave ϕ invariant is just

$$(\rho_i)_{ab}(xg_Ax^{-1})\phi(g^{-1}x) = \delta_{ab}\phi(x)\,. \tag{6.13}$$

If we take the trace of the left and right hand side of this equation, we obtain

$$\chi_{\rho_i}(g_A)\phi(g^{-1}x) = d_{\rho_i}\phi(x)\,. \tag{6.14}$$

Here, we have used the invariance of χ_{ρ_i} under conjugation to remove the conjugation with x. This equation is equivalent to (6.12), so any solution to (6.13) satisfies (6.12). The converse is also true. From (6.12), we see that ϕ has to be an eigenvector of the action of g^{-1} with eigenvalue $\frac{\chi_{\rho_i}(g_A)}{d_{\rho_i}}$. Since g has finite order, this implies that $\frac{\chi_{\rho_i}(g_A)}{d_{\rho_i}}$ is a root of unity. This means that $\rho_i(g_A)$ must be $\frac{\chi_{\rho_i}(g_A)}{d_{\rho_i}}$ times the unit matrix, since $\chi_{\rho_i}(g_A)$ is the sum of the eigenvalues of $\rho_i(g_A)$, which are all roots of unity (ρ_i is unitary). But if this holds, then $\rho_i(xg_Ax^{-1})$ is also $\frac{\chi_{\rho_i}(g_A)}{d_{\rho_i}}$ times the unit matrix and hence (6.13) is satisfied.

Using that the coproduct of $D(H)^*$ corresponds to the product of $D(H)$ and also that Π^A_α and ϵ are algebra homomorphisms, one may easily show that the set of irreps (ρ_i, g) whose matrix elements solve (6.13) (or 6.12) closes under tensor products. It also clearly closes under conjugation. Therefore, $\mathcal{T}$ is spanned by the matrix elements of those irreps.

If, for all irreps (ρ_i, g) whose matrix elements span $\mathcal{T}$, we have $\frac{\chi_\rho(g_A)}{d_\rho} = 1$, then all the ρ_i are paired up with the same set of elements g of H, namely those elements whose action leaves ϕ invariant. These elements form a subgroup N_ϕ of H. IN this situation, $\mathcal{T}$ is the transformation group algebra $F(H/K) \otimes N_\phi \subset D(H)$, where K is the intersection of the kernels of the ρ_i. If one of the roots of unity $\frac{\chi_{\rho_i}(g_A)}{d_\rho}$ does not equal 1, then the representation (ρ_i, e) of $D(H)^*$ does not occur in $\mathcal{T}$, but (ρ_i, g) does, for some $g \neq e$. Hence $\mathcal{T}$ cannot be a transformation group algebra of the form $F(H/K) \otimes N$ in this case. □

7. Confinement

7.1 Confinement and Hopf quotients

As we have seen, the formation of a condensate of particles in the state ϕ breaks the Hopf symmetry $\mathcal{A}$ of a theory down to the Hopf stabilizer $\mathcal{T} \subset \mathcal{A}$ of ϕ. The particles in the effective theory which has the condensate as its ground state will thus carry irreducible representations of $\mathcal{T}$. However, not all the particles in the effective theory will occur as free particles; some will be confined. The intuition behind this is simple: if a particle in the effective theory has non-trivial monodromy with the condensate particles, then it will “draw a string” in the condensate. That is, the condensate’s order parameter has to have a (half)line discontinuity as a consequence of the non-trivial parallel transport around the location of the particle. This line discontinuity corresponds physically to a domain wall and will cost a fixed amount of energy per unit of length[4] and hence it may not extend to infinity. As a consequence, single particles that have non-trivial braiding with the condensate cannot occur. On the other hand, configurations such as a particle and its antiparticle connected by a (short) finite length string may occur and we may compare these to the mesons of QCD. Similarly, one may have baryon-like excitations, which are bound states of three or more elementary excitations which do not match in pairs. Thus, we expect all the irreps of the Hopf stabilizer of the condensate to occur as particles in the broken theory, but some of them will occur as free particles, while others will occur only as constituents of mesonic or baryonic excitations.

There are some requirements which should hold for the set of representations of $\mathcal{T}$ that do not get confined. Clearly, this set should contain the vacuum representation or counit of T. Also, it should be closed under tensor products and charge conjugation; we would not want two non-confined particles to fuse to a confined particle, and, given that a particle is not confined, we would like the same to hold for its charge-conjugate. From these conditions on the non-confined representations it follows, using the results of section 6.2, that the matrix elements of the representations of the non-confined irreps of $\mathcal{T}$ span a Hopf-subalgebra of $\mathcal{T}^*$. We will call this subalgebra $\mathcal{U}^*$. Again using the results in section 6.2, it follows that the dual $\mathcal{U}$ of $\mathcal{U}^*$ will be a Hopf algebra whose irreducible representations are exactly the representations of $\mathcal{T}$ which are not confined (and whose matrix elements span $\mathcal{U}^*$). The dual map of the embedding of $\mathcal{U}^*$ into $\mathcal{T}^*$ is a Hopf map from $\mathcal{T}$ onto $\mathcal{U}$ and therefore $\mathcal{U}$ is a Hopf quotient of $\mathcal{T}$. This Hopf quotient $\mathcal{U}$ may be seen as the symmetry which classifies the non-confined excitations of the system. A schematic picture of the main symmetry algebras defined in this paper and their relations may be found on page 57.

To determine which irreps of $\mathcal{T}$ correspond to free particles and which are confined, we need to have a notion of braiding between a representation π of the original Hopf algebra and a representation ρ of $\mathcal{T}$. Clearly, the braiding should be derived from the R-matrix of $\mathcal{A}$. Let us write this as $R = \sum_k R_k^1 \otimes R_k^2$. Unfortunately, we cannot just define the matrix for an exchange of a ρ and a π as $\sigma \circ (\rho \otimes \pi)(R)$, since the R_k^1 are not usually elements of $\mathcal{T}$. However, we can take the exchange matrix to be $\sigma \circ ((\rho \circ P) \otimes \pi)(R)$, where P is the orthogonal projection of $\mathcal{A}$ onto $\mathcal{T}$. We also define the braid matrix for the product $\pi \otimes \rho$ as $\sigma \circ (\pi \otimes (\rho \circ P))(R)$. A representation ρ of $\mathcal{T}$ should now correspond to a free particle excitation if these braiding matrices have trivial action on the product of the condensate vector with an arbitrary vector in the module of ρ. That is, for a non-confined representation ρ, we would like to demand

$$\sum_k \rho(P(R_k^1)) \otimes \pi(R_k^2)\phi = \rho(1) \otimes \phi$$

[4]Note that we have not specified the hamiltonian in our model, but we assume here that, behind the scenes, there is a “Higgs potential” which causes the symmetry breaking condensation. Such a potential will make strings cost an amount of energy that increases linearly with their length.

$$\sum_k \pi(R_k^1)\phi \otimes \rho(P(R_k^2)) \;=\; \phi \otimes \rho(1)\,. \tag{7.1}$$

This gives us a requirement on every matrix element of each of the non-confined representations ρ. These matrix elements are of course elements of $\mathcal{T}^*$ and we may in fact write down a corresponding requirement for arbitrary elements of $\mathcal{T}^*$. To do this, we first define a left and a right action of $\mathcal{T}^*$ on the module V_π of the representation π of $\mathcal{A}$. We take

$$\begin{aligned} f \cdot v \;&:=\; \sum_k f(P(R_k^1))\pi(R_k^2)v \\ v \cdot f \;&:=\; \sum_k f(P(R_k^2))\pi(R_k^1)v\,, \end{aligned} \tag{7.2}$$

where $f \in \mathcal{T}^*$ and $v \in V_\pi$.

Proposition 7. *Let $\star$ denote the multiplication on $\mathcal{T}^*$. Then*

$$\begin{aligned} (f_1 \star f_2) \cdot v \;&=\; f_1 \cdot (f_2 \cdot v) \\ v \cdot (f_1 \star f_2) \;&=\; (v \cdot f_1) \cdot f_2\,. \end{aligned} \tag{7.3}$$

Proof. We have

$$\begin{aligned} (f_1 \star f_2) \cdot v \;&=\; (f_1 \otimes f_2 \otimes \pi) \circ (\Delta \otimes \mathrm{id}) \circ (P \otimes \mathrm{id})(R)v \\ &=\; (f_1 \otimes f_2 \otimes \pi) \circ (P \otimes P \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})(R)v \\ &=\; (f_1 \otimes f_2 \otimes \pi) \circ (P \otimes P \otimes \mathrm{id})(R_{13}R_{23})v \\ &=\; f_1(P(R_l^1))f_2(P(R_k^1))\pi(R_l^2 R_k^2) \\ &=\; f_1(P(R_l^1))\pi(R_l^2)f_2(P(R_k^1))\pi(R_k^2)v \\ &=\; f_1 \cdot (f_2 \cdot v)\,. \end{aligned} \tag{7.4}$$

In the fourth equality, we used that π is a representation of $\mathcal{A}$. In the third equality, we used $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$. In the second equality, we used the fact that the orthogonal projection P commutes with the comultiplication, that is

$$(P \otimes P) \circ \Delta = \Delta \circ P\,. \tag{7.5}$$

One may see that this holds by evaluating both sides on the basis of $\mathcal{A}$ that is given by the matrix elements of the irreps of $\mathcal{A}^*$. Using this property of P and $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$, one may similarly prove that $v \cdot (f_1 \star f_2) = (v \cdot f_1) \cdot f_2$. $\square$

The requirements (7.1) on the matrix elements of ρ may now be generalized to

$$f \cdot \phi = \phi \cdot f = f(1)\phi = \epsilon^*(f)\phi\,. \tag{7.6}$$

Hence, the requirement that the representation ρ has trivial braiding with the condensate becomes the requirement that the left and right action of the matrix elements of ρ, as defined above, leave the condensate invariant (in the sense of definition 4). Thus, we may say that passing from the unbroken symmetry $\mathcal{T}$ to the unconfined symmetry $\mathcal{U}$ is equivalent to breaking the dual $\mathcal{T}^*$ down to $\mathcal{U}^*$ by the condensate ϕ. If we take this point of view, then the fact that we are talking about braiding is hidden in the definition of the actions above.

Unfortunately, it turns out that (7.6) does not always have solutions. In particular, the counit $\epsilon_{\mathcal{T}}$ of $\mathcal{T}$ does not always solve (7.6) (or (7.1)). This is linked to the fact that the "action" we defined above preserves the multiplication, but not necessarily the unit $\epsilon_{\mathcal{T}}$ of $\mathcal{T}^*$.

Therefore, to ensure that $\mathcal{U}^*$ contains at least the "vacuum representation" $\epsilon_{\mathcal{T}}$ of $\mathcal{T}^*$, we change the condition (7.6) to

$$\begin{aligned} f \cdot \phi &= f(1)\epsilon \cdot \phi \\ \phi \cdot f &= f(1)\phi \cdot \epsilon \,. \end{aligned} \tag{7.7}$$

In other words, we no longer demand that the elements of $\mathcal{U}^*$ leave the condensate invariant, but instead, we ask that they act on the condensate in the same way as ϵ. To put it yet another way, we say that the representations of $\mathcal{T}$ that are not confined are those representations that have the same braiding with the condensate as the vacuum representation. Note that, since $\epsilon_{\mathcal{T}} = 1_{\mathcal{T}^*}$, we may also write the above condition as

$$\begin{aligned} f \cdot (\epsilon \cdot \phi) &= \epsilon^*(f)(\epsilon \cdot \phi) \\ (\phi \cdot \epsilon) \cdot f &= \epsilon^*(f)(\phi \cdot \epsilon) \,. \end{aligned} \tag{7.8}$$

We may thus still see confinement as a dual symmetry breaking, but now $\mathcal{T}^*$ is not broken to $\mathcal{U}^*$ by the original condensate ϕ, but by the vectors $\epsilon \cdot \phi$ and $\phi \cdot \epsilon$. Clearly, when $\epsilon \cdot \phi = \phi$ and $\phi \cdot \epsilon = \phi$, the new condition on elements of $\mathcal{U}^*$ reduces to (7.6).

We have now defined an algebra $\mathcal{U}$ (through its dual $\mathcal{U}^*$) whose representations should classify the non-confined excitations over the condensate. There should be an action of the braid group on the Hilbert space for a number of such excitations. Therefore, we should like $\mathcal{U}$ to be quasitriangular with an R-matrix related to the R-matrix of the original Hopf algebra $\mathcal{A}$ (for example $(P \otimes P)(R_{\mathcal{A}})$). As we will see in the examples, the conditions (7.7) are often enough to ensure that $\mathcal{U}$ has such a quasitriangular structure. Nevertheless, it does not always seem to the case (this will be made somewhat clearer in section 11). Therefore we expect that the requirements (7.7) will in general have to be supplemented by some extra condition and the non-confined algebra could then be smaller than the algebra $\mathcal{U}$ defined here.

7.2 Domain walls and Hopf kernels

In the previous section, we talked about confined particles pulling strings in the condensate. These were line discontinuities in the condensate's wave function, induced by the non-trivial parallel transport around the confined particle. Evidently, the internal state of the condensate particles on one side of such a string will differ from that on the other side. Therefore, we may also view these strings as domain walls between regions with different condensates which exhibit the same symmetry breaking pattern.[5] We would like to classify such walls. Clearly, a wall is uniquely determined by the confined particles on which it may end, or in other words, by a representation of the residual algebra $\mathcal{T}$ that does not correspond to a representation of its non-confined quotient $\mathcal{U}$. However, there may be several irreps of $\mathcal{T}$ that cause the same parallel transport in the condensate and these will all pull the same string (or wall). In fact, let ρ be an irrep of $\mathcal{T}$ and let τ be a non-confined irrep of $\mathcal{T}$, then any irrep of $\mathcal{T}$ in the tensor product representation $(\rho\otimes\tau)\circ\Delta$ will pull the same string as ρ, since the non-confined irrep τ has trivial braiding with the condensate. In short, we may say that walls are unaffected by fusion with non-confined particles.

In view of the above, we expect that the wall that corresponds to a $\mathcal{T}$-representation ρ is already determined by the restriction of ρ to a subalgebra $\mathcal{W}$ of $\mathcal{T}$. This subalgebra should be such that, if τ is a non-confined irrep of $\mathcal{T}$, then the restriction to $\mathcal{W}$ of the tensor product representation $(\rho \otimes \tau) \circ \Delta$ should be isomorphic to a direct sum of copies of the restriction of ρ to $\mathcal{W}$ (the number of copies being the dimension of τ). Now it turns out that such a $\mathcal{W} \subset \mathcal{T}$ exists, and in fact, there are two logical options. Denote the Hopf map from the residual symmetry algebra $\mathcal{T}$ onto the non-confined algebra $\mathcal{U}$ by Γ. The *left Hopf kernel* $\mathrm{LKer}(\gamma)$ of Γ is then the subset of $\mathcal{T}$ defined as

$$\mathrm{LKer}(\Gamma) := \{t \in \mathcal{T} \,|\, (\Gamma \otimes \mathrm{id}) \circ \Delta(t) = 1_{\mathcal{U}} \otimes t\} \tag{7.9}$$

and similarly, the *right Hopf kernel* of Γ is defined as

$$\mathrm{RKer}(\Gamma) := \{t \in \mathcal{T} \,|\, (\mathrm{id} \otimes \Gamma) \circ \Delta(t) = t \otimes 1_{\mathcal{U}}\}\,. \tag{7.10}$$

These are our two candidates for $\mathcal{W}$. One may check that the left Hopf kernel is a right coideal subalgebra (that is, $\mathrm{LKer}(\Gamma)$ is a subalgebra and $\Delta(\mathrm{LKer}(\Gamma)) \subset \mathcal{T} \otimes \mathrm{LKer}(\Gamma)$) and similarly that the right Hopf kernel is a left coideal subalgebra. Moreover, one has $S(\mathrm{RKer}(\Gamma)) = \mathrm{LKer}(\Gamma)$ and $S(\mathrm{LKer}(\Gamma)) = \mathrm{RKer}(\Gamma)$. Thus, $\mathrm{LKer}(\Gamma)$ is a Hopf subalgebra of $\mathcal{T}$ exactly if $\mathrm{LKer}(\Gamma) = \mathrm{RKer}(\Gamma)$. In the examples we will meet, this will not usually be the case. Note that, even if $\mathrm{LKer}(\Gamma) \neq \mathrm{RKer}(\Gamma)$, the representations of $\mathrm{LKer}(\Gamma)$ and $\mathrm{RKer}(\Gamma)$ are in one to one correspondence: the representation ρ of $\mathrm{LKer}(\Gamma)$ corresponds to the representation $\bar{\rho} := \rho^t \circ S$ of $\mathrm{RKer}(\Gamma)$. If $\mathrm{LKer}(\Gamma)$ and $\mathrm{RKer}(\Gamma)$ are semisimple algebras (which we will assume), then this isomorphism of representations induces an isomorphism of algebras and so $\mathrm{LKer}(\Gamma) \cong \mathrm{RKer}(\Gamma)$. In other words: our candidates for $\mathcal{W}$ are isomorphic and it does not really matter which one we take.

Conjecture 1. *The wall corresponding to the $\mathcal{T}$-irrep Ω^B_β is charcterized by the restriction of Ω^B_β to either* $\mathrm{LKer}(\Gamma)$ *or* $\mathrm{RKer}(\Gamma)$.

We provide the following evidence for this conjecture

1. If ρ is a representation of $\mathrm{LKer}(\Gamma)$ and τ is a representation of $\mathcal{T}$, then we define the tensor product as $(\tau \otimes \rho) \circ \Delta$. This is well-defined, since $\mathrm{LKer}(\Gamma)$ is a right coideal of

[5] One may in principle also have domain walls between condensates with different symmetry breaking patterns, but this requires the parameters which govern the system (the symmetry breaking potential) to vary as one crosses the walls, breaking translational symmetry at the level of the lagrangian or hamiltonian

$\mathcal{T}$. It is also clearly a representation of LKer(Γ). Now suppose that τ corresponds to a representation $\tilde{\tau}$ of $\mathcal{U}$, that is, $\tau = \tilde{\tau} \circ \Gamma$. Then, the defining property of LKer(Γ) guarantees that we have

$$(\tau \otimes \rho) \circ (\Gamma \otimes \mathrm{id}) \circ \Delta(t) = \tau(1) \otimes \rho(t) \tag{7.11}$$

for all $t \in \mathrm{LKer}(\Gamma)$. In other words, LKer(Γ) is indeed defined in such a way that its representations are not affected by fusion with representations of the non-confined algebra $\mathcal{U}$, just as walls are not affected by fusion with non-confined particles. Clearly, we may also define the tensor product $(\bar{\rho} \otimes \tau) \circ \Delta$ of an RKer(Γ) representation $\bar{\rho}$ with a representation τ of $\mathcal{T}$ and again, the fusion will be trivial if τ corresponds to a representation of $\mathcal{U}$.

2. Every representation of $\mathcal{T}$ corresponds to a representation of LKer(Γ) by restriction. In particular, if ρ is an irrep of $\mathcal{T}$ which factors over Γ, that is $\rho = \tau \circ \Gamma$, with τ a representation of $\mathcal{U}$, then we have, for all $t \in \mathrm{LKer}(\Gamma)$:

$$\begin{aligned} \rho(t) &= \tau \circ \Gamma(t) \\ &= \tau \circ \Gamma \circ (\mathrm{id} \otimes \epsilon) \circ \Delta(t) \\ &= \tau \circ (\mathrm{id} \otimes \epsilon) \circ (\Gamma \otimes \mathrm{id}) \circ \Delta(t) = \epsilon(t)\tau(1)\,. \end{aligned} \tag{7.12}$$

Thus, we see that the non-confined irreps of $\mathcal{T}$ all correspond to the trivial representation of LKer(Γ). This result is consistent with the fact that the non-confined representations of $\mathcal{T}$ do not pull strings. Again, a similar result holds for RKer(Γ).

3. If B is a finite dimensional Hopf algebra, C a Hopf quotient of B and A the corresponding left Hopf kernel, then it is known by a theorem of Schneider (theorem 2.2 in [39], see also [14] for background) that B is isomorphic to a crossed product of A and C as an algebra and also as a left A-module and as a right C-comodule. Such crossed products are defined as follows:

Definition 8. *Let C be a Hopf algebra, let A be an algebra and let $\sigma : C \otimes C \to A$ be a convolution-invertible linear map. Also, suppose we have a linear map from $B \otimes A$ to A, which we write as $b \otimes a \mapsto b \cdot a$. We require that A is a twisted B-module, that is, $1 \cdot a = a$ for all a and*

$$c \cdot (d \cdot a) = \sum \sigma(c_1, d_1)(c_2 d_2 \cdot a)\sigma^{-1}(c_3, d_3) \tag{7.13}$$

Here we use Sweedler notation for the coproduct. We also require that σ is a cocycle, that is

$$\begin{aligned} &\sigma(c, 1) = \sigma(1, c) = \epsilon(c)1 \\ &\sum (c_1) \cdot \sigma(d_1, e_1))\, \sigma(c_2, d_2 e_2) = \sum \sigma(c_1, d_1)\sigma(c_2 d_2, e) \end{aligned} \tag{7.14}$$

and that C measures A:

$$c \cdot 1 = \epsilon(c)1\,, \qquad c \cdot (ab) = \sum (c_1 \cdot a)(c_2 \cdot b)\,. \tag{7.15}$$

Now the crossed product algebra $A\#_\sigma B$ *is the vector space* $A \otimes B$ *with the product given by*

$$(a \otimes c)(b \otimes d) = \sum a(c_1 \cdot b)\sigma(c_2, d_1) \otimes c_3 d_2 \,. \qquad (7.16)$$

These crossed products were introduced in [40, 41]. An accessible treatment may be found in [14] or [18]. When it is given that C measures A, one may show that the conditions that involve σ are equivalent to the associativity of the product of $A\#_\sigma C$. Some elementary properties of the crossed product are

- A is embedded into $A\#_\sigma C$ through $a \mapsto a \otimes 1$, that is, we have

$$(a \otimes 1)(b \otimes 1) = (ab \otimes 1)\,.$$

- The map $j : C \to A\#_\sigma C$ given by $c \mapsto 1 \otimes c$ is clearly a C-comodule morphism when $A\#_\sigma C$ and C are given the comodule structures $\mathrm{id}_A \otimes \Delta_C$ and Δ_C respectively, but it is usually not an algebra morphism; we have $(1 \otimes c)(1 \otimes d) = \sum(\sigma(c_1, d_1) \otimes c_2 d_2)$.

- When σ is trivial, that is $\sigma(c, d) = \epsilon(c)\epsilon(d)$, the cross product is just the ordinary smash product; we have $(a \otimes b)(c \otimes d) = \sum a(c_1 \cdot b) \otimes c_2 d_2$. In this case, j is an algebra morphism.

We see that our residual algebra $\mathcal{T}$ is isomorphic to the cross product $\mathrm{LKer}(\Gamma)\#_\sigma\mathcal{U}$ for some cocycle σ. This lends support to the idea that $\mathcal{T}$-excitations are characterized by a wall, corresponding to a representation of $\mathrm{LKer}(\Gamma)$ and by further quantum numbers, which can be associated to the non-confined algebra $\mathcal{U}$. If the cross product was just the tensor product $\mathrm{LKer}(\Gamma) \otimes \mathcal{U}$, then these "non confined quantum numbers" would be labels of $\mathcal{U}$-representations, but here, we cannot expect this, because the actions of $\mathrm{LKer}(\Gamma)$ and $\mathcal{U}$ on a $\mathcal{T}$-module will not commute. In fact, $\mathcal{U}$ is typically not even a subalgebra of $\mathcal{T}$. Therefore, finding the quantum numbers associated to $\mathcal{U}$ for the general case is a non-trivial task, which we postpone to future work.[6]

7.3 Confinement for transformation group algebras

Suppose the $D(H)$ symmetry of a discrete gauge theory has been broken by a condensate $\phi \in V_\alpha^A$ and the residual symmetry algebra $\mathcal{T}$ is a transformation group algebra of the kind referred to in section 6.3. The explicit definition is

$$\mathcal{T} = \{F \in D(H) | F(xk, y) = F(x, y)1_N(y)\ (\forall k \in K)\} \qquad (7.17)$$

where N is a subgroup of H and K is a normal subgroup of H. Such algebras will arise frequently in our examples. Here, we investigate which representations of such a $\mathcal{T}$ are confined and which are not. In particular, we will find that there is a set of non-confined representations of $\mathcal{T}$ such that the irreps in this set are in one to one correspondence with those of $D(N/(N \cap K))$.

[6] Note that much more than we have written here is known when $\mathrm{LKer}(\Gamma)$ is a Hopf algebra. For such results, see for instance [42, 43, 44].

First, we find some properties of the condensate vector ϕ. If $\mathcal{T}$ is of the form given above then the invariance of ϕ under elements of the form $1 \otimes n$ (with $n \in N$) and $f \otimes e$ implies that we have

$$\begin{aligned} (\forall n \in N): \quad \phi(nx) &= \phi(x) \\ (\forall (f \otimes e) \in \mathcal{T}): \quad f(xg_A x^{-1})\phi(x) &= f(e)\phi(x) \,. \end{aligned} \tag{7.18}$$

The second equation and $\phi \neq 0$ imply that there is an $x \in H$ such that $f(xg_A x^{-1}) = f(e)$ for all f that are constant on K-cosets. As a consequence, we have $xg_A x^{-1} \in K$ and hence, since K is normal in H, we have $A \subset K$ and in particular $g_A \in K$.

Now let us write down an explicit formula for the orthogonal projection P of $D(H)$ onto $\mathcal{T}$:

$$P(F)(x,y) = \frac{1}{|K|}\int_K \mathrm{d}k\, F(xk,y) 1_N(y) \,. \tag{7.19}$$

In the following, we will sometimes omit the characteristic function of N and just keep in mind that the projected function has support in N. With the above formula, we can find $(P \otimes \mathrm{id})(R)$ and $(\mathrm{id} \otimes P)(R)$ from the formula (4.2) for the R-matrix of $D(H)$:

$$\begin{aligned} (P \otimes \mathrm{id})(R)(x_1,y_1,x_2,y_2) &= \frac{1}{|K|}\int_K \mathrm{d}k\, \delta_e(x_1 k (y_2)^{-1})\delta_e(y_1) \\ (\mathrm{id} \otimes P)(R)(x_1,y_1,x_2,y_2) &= \delta_e(x_1(y_2)^{-1})\delta_e(y_1)1_N(y_2) \,. \end{aligned} \tag{7.20}$$

Using these formulae, we can write down the left and right actions of $\mathcal{T}^*$ on the condensate vector, as defined in equation (7.2). They are given by

$$\begin{aligned} (\tau \cdot \phi)(x) &= \int_H \mathrm{d}z\, (\tau(F_L))(z)\phi(z^{-1}x) \\ (\phi \cdot \tau)(x) &= (\tau(F_R))\phi(x) \,, \end{aligned} \tag{7.21}$$

where τ is an arbitrary element of $\mathcal{T}^*$ and we have defined

$$\begin{aligned} F_L(a,b\,;z) &= \frac{1}{|K|}\int_K \mathrm{d}k\, \delta_e(akz^{-1})\delta_e(b) \\ F_R(a,b\,;x,z) &= \delta_e(xg_A x^{-1} b^{-1})1_N(b) \,. \end{aligned} \tag{7.22}$$

F_R and F_L should be read as functions of a and b with parameters x and z. We want to find the maximal Hopf subalgebra of $\mathcal{T}^*$ for which the condition (7.7) holds. This will be spanned (as a linear space) by the matrix elements of a set of representations of $\mathcal{T}$. Since $\mathcal{T}$ is isomorphic to a transformation group algebra, we know its representations (see section 4.2). They are labeled by an orbit B of the action of N on H/K and by an irreducible representation β of the stabilizer $N_B \subset N$ of this orbit. The matrix elements of the representation labeled by B and β in the basis of formula (4.11) can be read off from formula (4.12), which in this case becomes

$$\tau^B_\beta(F)^{i,j}_{\zeta,\eta} = \int_{N_B} F\left(x_\eta g_B x_\eta^{-1}, x_\eta n x_\zeta^{-1}\right)\beta_{i,j}(n)\,\mathrm{d}n \,. \tag{7.23}$$

In this formula, we have $x_\eta, x_\zeta \in N$ as in (4.12), while g_B is an arbitrary element of the K-coset ξ_B that features in (4.12). Note that it does not matter which element of this coset we take, since the function F in the integrand is constant on K-cosets in its left argument.

Proposition 8. *The requirements (7.7) which determine which of the irreps τ_β^B of $\mathcal{T}$ are not confined, reduce to*

$$(\forall\eta\in\mathcal{O}_B): \frac{1}{|K|}\int_K \mathrm{d}k\,\phi\left(kx_\eta g_B^{-1}x_\eta^{-1}x\right) = \frac{1}{|K|}\int_K \mathrm{d}k\,\phi(kx) \tag{7.24}$$

$$(g_A\notin N)\vee\left(\forall x\in\mathrm{supp}(\phi),\,\forall\eta\in\mathcal{O}_B:\beta(x_\eta^{-1}xg_Ax^{-1}x_\eta)=I\right). \tag{7.25}$$

Here, $\mathcal{O}_B$ is the orbit of ξ_B in H/K and I is the unit matrix in the module of β.

Proof. Substituting (7.23) into to the formulae for the actions above, we find for the left action

$$\begin{aligned}\left((\tau_\beta^B)_{\zeta,\eta}^{i,j}\cdot\phi\right)(x) &= \frac{1}{|K|}\int_H \mathrm{d}z\int_{N_B}\mathrm{d}n\int_K \mathrm{d}k\,\delta_e(x_\eta g_B x_\eta^{-1}kz^{-1})\,\delta_e\left(x_\eta n x_\zeta^{-1}\right)\beta_{i,j}(n)\phi(z^{-1}x)\\ &= \frac{1}{|K|}\int_{N_B}\mathrm{d}n\int_K \mathrm{d}k\,\delta_e\left(x_\eta n x_\zeta^{-1}\right)\beta_{i,j}(n)\phi\left(k^{-1}x_\eta g_B^{-1}x_\eta^{-1}x\right)\\ &= 1_{N_B}\left(x_\eta^{-1}x_\zeta\right)\beta_{i,j}\left(x_\eta^{-1}x_\zeta\right)\frac{1}{|K|}\int_K \mathrm{d}k\,\phi\left(kx_\eta g_B^{-1}x_\eta^{-1}x\right)\end{aligned} \tag{7.26}$$

and similarly, for the right action

$$\begin{aligned}(\phi\cdot(\tau_\beta^B)_{\zeta,\eta}^{i,j})(x) &= \int_{N_B}\mathrm{d}n\,\delta_e(xg_Ax^{-1}x_\zeta n^{-1}x_\eta^{-1})\beta_{i,j}(n)\phi(x)\\ &= 1_{N_B}(x_\eta^{-1}xg_Ax^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}xg_Ax^{-1}x_\zeta)\phi(x)\,.\end{aligned} \tag{7.27}$$

As a special case, we can find the left and right action of the counit $\epsilon\in\mathcal{T}^*$, which corresponds to the one-dimensional representation $\tau_1^{[e]}$. We have

$$\begin{aligned}(\epsilon\cdot\phi)(x) &= \frac{1}{|K|}\int_K \mathrm{d}k\,\phi(kx)\\ (\phi\cdot\epsilon)(x) &= 1_N(g_A)\phi(x)\,.\end{aligned} \tag{7.28}$$

The final ingredient we need in order to write down the requirements (7.7) for the matrix elements, is the value of $(\tau_\beta^B)_{\zeta,\eta}^{i,j}$ in $1_\mathcal{T}=1_{D(H)}$. This is given by

$$(\tau_\beta^B)_{\zeta,\eta}^{i,j}(1_{D(H)}) = 1_{N_B}(x_\eta^{-1}x_\zeta)\beta_{ij}(x_\eta^{-1}x_\zeta). \tag{7.29}$$

Thus, the conditions (7.7) that the matrix elements of τ_β^B have to fulfill, in order for τ_β^B not to be confined, become

$$1_{N_B}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\frac{1}{|K|}\int_K \mathrm{d}k\,\phi(kx_\eta g_B^{-1}x_\eta^{-1}x) = 1_{N_B}(x_\eta^{-1}x_\zeta)\beta_{ij}(x_\eta^{-1}x_\zeta)\frac{1}{|K|}\int_K \mathrm{d}k\,\phi(kx) \tag{7.30}$$

and

$$1_{N_B}(x_\eta^{-1}xg_Ax^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}xg_Ax^{-1}x_\zeta)\phi(x) = 1_N(g_A)1_{N_B}(x_\eta^{-1}x_\zeta)\beta_{ij}(x_\eta^{-1}x_\zeta)\phi(x)\,. \tag{7.31}$$

In the special case where $\eta=\zeta$ and $i=j$, the condition (7.30) reduces to (7.24). On the other hand, if (7.24) holds, then (7.30) will also hold for general (η,ζ,i,j) and hence (7.24) is equivalent to (7.30). The condition (7.31) is trivially satisfied when g_A is not contained in N (this is the first alternative in (7.25)). When g_A is an element of N (and hence of $N \cap K$, using (7.18)), it may also be simplified; in the special case where $\eta = \zeta$, (7.31) reduces to

$$1_{N_B}\left(x_\eta^{-1} x g_A x^{-1} x_\eta\right) \beta_{i,j}\left(x_\eta^{-1} x g_A x^{-1} x_\eta\right) \phi(x) = \delta_{ij} \phi(x)\,. \tag{7.32}$$

Now since $g_A \in N \cap K$, it follows that $x_\eta^{-1} x g_A x^{-1} x_\eta \in N \cap K$. But $N \cap K$ acts trivially on H/K and hence $N \cap K \subset N_B$ for any B. Hence the condition above reduces to the second alternative in (7.25). In the other direction, it is not difficult to see that (7.31) will be satisfied for general (η, ζ, i, j)if (7.25) is satisfied. Thus, we see that (7.31) is equivalent to (7.25). □

Proposition 8 indicates how far we can go towards the general solution of (7.7) without specifying the condensate vector ϕ. The following proposition describes a set of solutions that is present for any ϕ, but that is not always the full set of solutions.

Proposition 9. *Independently of the condensate vector ϕ, there is a set of unconfined irreps of $\mathcal{T}$ which closes under conjugation and tensor products. The corresponding Hopf quotient of $\mathcal{T}$ is isomorphic to the quantum double of the group $N/(N \cap K)$. This quantum double may be realized naturally on the space of functions on $N \times N$ which are constant on $(N \cap K)$-cosets in both arguments. The Hopf surjection $\Gamma : \mathcal{T} \to D(N/(N \cap K))$ is then given by*

$$\Gamma(f)(x, y) = \int_{N \cap K} f(x, yk) dk\,. \tag{7.33}$$

Proof. First, we find our set of irreps. Note that the left hand side of (7.24) is $\frac{1}{|K|}$ times the sum of the values of ϕ over the K-coset $x_\eta g_B^{-1} x_\eta^{-1} x K$. Similarly, the right hand side involves a sum over xK. Using the fact that $\phi(nx) = \phi(x)$ for any $n \in N$ (cf. (7.18)), we see that (7.24) will be satisfied if $x_\eta g_B^{-1} x_\eta^{-1} x K = nxK$ for some $n \in N$, or equivalently if there is an $n \in N$ such that $g_B K = nK$. Furthermore, (7.25) is clearly satisfied for all β that are trivial on $N \cap K \subset N_B$. Thus, the irreps τ_β^B of $\mathcal{T}$ for which $g_B K = nK$ and $\beta|_{N \cap K} = 1$ are never confined.

Second, we show that these irreps are in one-to one correspondence with the irreps of $D(N/(N \cap K))$. To see this, first note that, for $n_1, n_2 \in N$, we have

$$n_1 K = n_2 K \Leftrightarrow n_1(K \cap N) = n_2(K \cap N)\,. \tag{7.34}$$

In fact, let $\bar{N}$ denote the subgroup of H/K which consists of the classes nK with $n \in N$, then this correspondence is an isomorphism between $\bar{N}$ and $N/(N \cap K)$. It follows that the N-orbits in H/K whose elements lie in $\bar{N}$ are in one to one correspondence with the conjugacy classes of $N/(K \cap N)$. Now fix an arbitrary such orbit $B \subset \bar{N}$. The irreps β of the stabilizer $N_B \subset N$ of this orbit which are trivial on $K \cap N$ are in one to one correspondence with the irreps of $N_B/(K \cap N)$. But $N_B/(K \cap N)$ is exactly the centralizer of the conjugacy class of $N/(K \cap N)$ that corresponds to B. Hence the non-confined irreps of T_A are labeled by a conjugacy class of $N/(K \cap N)$ and an irrep of the centralizer of this class in $N/(K \cap N)$. But this means that they are in one to one correspondence with the irreps of $D(N/(K \cap N))$.

Now let us have a closer look at the map $\Gamma : \mathcal{T} \to D(N/(N\cap K))$. For convenience, we will realize $D(N/(N\cap K))$ on the space of functions on $N\times N$ which are constant on $(N\cap K)$-cosets in both arguments. The isomorphism with the usual formulation in terms of functions on $N/(N\cap K)$ is taken as follows. Let $\bar{f} \in F(N/(N\cap K)\times N/(N\cap K))$ then $\bar{f}$ corresponds to the function $f\in F(N\times N)$ given by $f(x,y) = \bar{f}(x(N\cap K), y(N\cap K))$. The demand that this identification is an isomorphism fixes the Hopf algebra structure on $F(N\times N)$. For example, the product of two functions on $N\times N$ may now be written as

$$f_1 \bullet f_2(x,y) = \frac{1}{|N\cap K|}\int_N f_1(x,z) f_2(z^{-1}xz, z^{-1}y)\,dz\,. \tag{7.35}$$

It is straightforward to prove that Γ, as defined above, is indeed a Hopf homomorphism. For example, to see that Γ preserves the product, we write

$$\begin{aligned}\Gamma(f_1)\bullet\Gamma(f_2)(x,y) &= \frac{1}{|N\cap K|}\int_N dz \int_{N\cap K} dk_1 \int_{N\cap K} dk_2\, f_1(x,zk_1) f_2(z^{-1}xz, z^{-1}yk_2)\\ &= \frac{1}{|N\cap K|}\int_N dz \int_{N\cap K} dk_1 \int_{N\cap K} dk_2\, f_1(x,z) f_2(z^{-1}xz, k_1 z^{-1}yk_2)\\ &= \int_N dz \int_{N\cap K} dk_2\, f_1(x,z) f_2(z^{-1}xz, z^{-1}yk_2) = \Gamma(f_1\bullet f_2)(x,y)\,.\end{aligned} \tag{7.36}$$

In going from the first to the second line, we used the invariance of the integral over N and the fact that f_2 is constant on K-cosets in its left argument. In going from the second to the third line, we used the invariance of the k_2-integral to remove the k_1-dependence from the integrand and subsequently removed the k_1-integral. The rest of the proof that Γ is a Hopf algebra morphism is similar. It is also easy to see that Γ is surjective.

To complete the proof of the proposition, we need to show that the set of unconfined irreps we have found consists precisely of those irreps of $\mathcal{T}$ that factor over Γ. Thus, let $\Pi^{\tilde{B}}_{\tilde{\beta}}$ be an irrep of $D(N/N\cap K)$ and let $\chi^{\tilde{B}}_{\tilde{\beta}}$ be its character, as given in (4.13). Then we have

$$\chi^{\tilde{B}}_{\tilde{\beta}}(\Gamma(f)) = \int_{\tilde{B}} d\zeta \int_{N_{\tilde{B}}} dn \int_{N\cap K} dk\, f\left(x_\zeta g_{\tilde{B}} x_\zeta^{-1}, x_\zeta n x_\zeta^{-1} k\right)\chi_{\tilde{\beta}}(n)\,. \tag{7.37}$$

Here, we have abused notation slightly: in stead of elements of $N/(N\cap K)$ one should read representatives of these elements in N where appropriate. It should be clear that the choice of representatives does not affect the result. We may now change the sum over the conjugacy class $\tilde{B}\subset N/(N\cap K)$ into a sum over the corresponding N-orbit $B\subset \bar{N}\subset H/K$. Similarly, we may change the sums over $N_{\tilde{B}}$ and $N\cap K$ into one sum over $N_B\subset N$. This yields

$$\chi^{\tilde{B}}_{\tilde{\beta}}(\Gamma(f)) = \int_B d\zeta \int_{N_B} dn\, f\left(x_\zeta g_B x_\zeta^{-1}, x_\zeta n x_\zeta^{-1}\right)\chi_\beta(n)\,, \tag{7.38}$$

where β is the irrep of N_B that corresponds to the irrep $\tilde{\beta}$ of $N_{\tilde{B}}$ (of course, β is trivial on $N\cap K$). The expression above is just the value on f of the character of the irrep τ^B_β of $\mathcal{T}$. τ^B_β indeed belongs to our set of unconfined irreps and from the one to one correspondence between irreps of $D(N/(K\cap N))$ and irreps in our unconfined set, we see that must get all irreps in the unconfined set in this way. Thus, the set of unconfined irreps of $\mathcal{T}$ that we have found corresponds precisely to the set of irreps of $\mathcal{T}$ that factor over Γ and the proposition follows. □

Note that the R-matrix and ribbon-element of $D(N/(K\cap N))$ provide the set of non-confined irreps that we have found above with a well defined braiding and spin. It is not clear that we will have such properties for the full set of solutions to (7.24) and (7.25). Therefore, we expect that the physically relevant set of solutions to these equations is the one given in the proposition above.This issue will not be very important in the sequel, since the set of solutions in the proposition is actually complete in all our examples.

Proposition 10. *The left and right Hopf kernels of* Γ *are given by*

$$\begin{aligned}\mathrm{LKer}(\Gamma) &= \{f\in\mathcal{T}|\forall x_1\in N\colon f(x_1x_2,y)=f(x_2,y)\wedge\forall y\notin N\cap K\colon f(x,y)=0\} \qquad (7.39)\\ \mathrm{RKer}(\Gamma) &= \{f\in\mathcal{T}|\forall x_2\in N\colon f(x_1x_2,y)=f(x_1,y)\wedge\forall y\notin N\cap K\colon f(x,y)=0\}\,. \qquad (7.40)\end{aligned}$$

Proof. We have

$$\begin{aligned}\Gamma\otimes\mathrm{id}(\Delta(f))(x_1,y_1,x_2,y_2) &= \int_{N\cap K} f(x_1x_2,y_1k)\delta_e(y_1ky_2^{-1})\\ (1_{D(N/(N\cap K))}\otimes f)(x_1,y_1,x_2,y_2) &= 1_{N\cap K}(y_1)f(x_2,y_2) \qquad (7.41)\end{aligned}$$

and the left Hopf kernel of Γ consists of those functions f for which the right hand sides of these equations are equal:

$$\int_{N\cap K} f(x_1x_2,y_1k)\delta_e(y_1ky_2^{-1}) = 1_{N\cap K}(y_1)f(x_2,y_2)\,. \qquad (7.42)$$

Now if we take $y_1=y_2$, then this requirement reduces to

$$f(x_1x_2,y_1) = 1_{N\cap K}(y_1)f(x_2,y_1)\,, \qquad (7.43)$$

from which we see that $f(x,y)$ equals zero for all $x\in H$ when y is not an element of $N\cap K$, while for $y\in N\cap K$, we have $f(x_1x_2,y)=f(x_2,y)$ for all $x_1\in N$. On the other hand, all f which satisfy these requirements automatically satisfy (7.42). One may see this by noting that both the left hand side and the right hand side of (7.42) can be non-zero only if both y_1 and y_2 are elements of $K\cap N$, in which case left hand side and right hand side are equal. The formula for $\mathrm{LKer}(\Gamma)$ now follows. The proof of the expression for $\mathrm{RKer}(\Gamma)$ is similar and we leave it to the reader. □

If we once again let $\bar{N}$ be the subgroup of H/K that consists of the cosets nK of the elements of N, then we see that we have the following

Corollary 3. *As algebras:*

$$\begin{aligned}\mathrm{LKer}(\Gamma) &\cong F(\bar{N}\backslash(H/K))\otimes\mathbb{C}(N\cap K)\\ \mathrm{RKer}(\Gamma) &\cong F\left((H/K)/\bar{N}\right)\otimes\mathbb{C}(N\cap K)\,. \qquad (7.44)\end{aligned}$$

Proof. To see that the isomorphisms are algebra isomorphisms, note that the elements of $1\otimes\mathbb{C}(K\cap N)$ commute with those of $F(\bar{N}\backslash(H/K))\otimes 1$ and $F((H/K)/\bar{N})\otimes 1$. This is because the elements of $F(H/K)\otimes 1$ already commuted with those of $1\otimes\mathbb{C}(K\cap N)$ in $\mathcal{T}$. □

As a consequence of this corollary, each irreducible representation of the left kernel is a product of an irrep of $F(\bar{N}\backslash(H/K))$ and an irrep of $N\cap K$. The irreps of $F(\bar{N}\backslash(H/K))$ are of course labeled by the elements of $\bar{N}\backslash(H/K)$ and hence each irrep of $\mathrm{LKer}(\Gamma)$ is labeled by an element of $\bar{N}\backslash(H/K)$ and an irrep of $N\cap K$. Similarly, each irrep of $\mathrm{RKer}(\Gamma)$ is labeled by an element of $(H/K)/\bar{N}$ and an irrep of $N\cap K$.

8. Requirements on condensates

Before we turn to the study of explicit examples of symmetry breaking and confinement, let us first motivate the choices of condensate vectors that we will use in our examples.

Up to now we have assumed that one may form a condensate of any kind of particle in the theory, in any internal state ϕ. However, if we want to have true Bose condensates, then we should demand that the state ϕ has trivial self-braiding and also trivial spin factor.[7] In other words:

- The condensate must have trivial spin factor, i.e $\alpha(g_A) = I$.
- The condensate must have trivial self-braiding, i.e.

$$\sigma \circ \Pi^A_\alpha \otimes \Pi^A_\alpha(R)\phi \otimes \phi = \phi \otimes \phi\,. \tag{8.1}$$

The examples that we will treat in the rest of this paper will all have trivial spin and self-braiding. The rest of this section is devoted to finding out which kinds of electric, magnetic and dyonic condensates will satisfy these requirements.

For any purely *electric* condensates $\phi \in V^e_\alpha$ (see section 9), the requirements are both trivially satisfied.

A vector ϕ in a purely *magnetic* $D(H)$-module V^A_1 will automatically have trivial spin, but may have non-trivial self-braiding. Nevertheless, there will always be at least two gauge orbits of magnetic states with trivial self-braiding for every class A which has more than a single element. The first of these orbits contains all the states with pure fluxes hg_Ah^{-1}, which have wave functions 1_{hN_A}. We will study the corresponding condensates in section 11. The second orbit, which will be studied in section 10, consists of the single gauge invariant state which is the superposition of all these pure fluxes. Its wave function is the function that sends all elements of H to 1. Of course if A has only a single element, then these orbits coincide. Note that, when the orbits are different, they will also have different symmetry breaking patterns. In particular, the gauge singlet will leave the electric group unbroken, while the states in the other orbit will not. To see that the states in the two orbits we have mentioned do indeed have trivial self-braiding and to see if there are more states with this property, we write down the expression for the self-braiding of an arbitrary $f \in V^A_1$. We have

$$\begin{aligned} f \otimes f \,:\, (g,h) &\mapsto f(g)f(h) \\ \sigma \circ R\,(f \otimes f) \,:\, (g,h) &\mapsto f(g)f(gg_A^{-1}g^{-1}h) \end{aligned} \tag{8.2}$$

and hence f has trivial self-braiding precisely when

$$f(g)f(h) = f(g)f(gg_A^{-1}g^{-1}h) \qquad (\forall g,h \in H)\,. \tag{8.3}$$

[7] In some applications, it could be more useful to think of our condensate as a background of particles in the same *internal* state, but not necessarily with the same external quantum numbers. Then the restrictions we give here are not necessary. Examples of “condensates” of particles with a non-trivial spin factor would be the fractional quantum Hall ground states proposed in [45, 46].

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One may readily check that the states we have already mentioned are always solutions to this equation. Depending on H and A, there may also be extra solutions. For example, if $A \subset N_A$, then all f are allowed, since in that case $hN_A = gg_A^{-1}g^{-1}hN_A$.

Dyons (see section 12)can have non-trivial spin, but dyons with trivial spin also exist for many groups H. In fact, given a magnetic flux A, there will be dyons with flux $A \neq [e]$ and trivial spin factor precisely when g_A is contained in a proper normal subgroup of its centralizer N_A. For abelian H, this just means that the cyclic group generated by g_A must be a proper subgroup of H. For non-abelian H, one may note that g_A is contained in the center of N_A, which is a proper normal subgroup if N_A is still non-abelian. If N_A is abelian, then we have the requirement that the cyclic group generated by g_A must be a proper subgroup of N_A. When H is abelian, the requirement of trivial self-braiding is equivalent to that of trivial spin and hence all the spinless dyons we have found may be condensed. When H is non-abelian, this is not the case and the requirement of trivial self-braiding then restricts the possibilities further. In particular, using the ribbon property of $D(H)$, it gives the necessary condition that two of the condensed dyons should be able to fuse into a particle with trivial spin. In spite of this restriction, there are still many non-abelian groups H which allow for dyonic states with trivial self-braiding. One may for example show that they occur for any non-abelian H with a non trivial center.

9. Electric condensates

9.1 Symmetry breaking

In this section, we study symmetry breaking by an electric condensate $\phi \in \Pi^e_\alpha$. The first thing to do is to find the residual symmetry algebra, which is the Hopf stabilizer of ϕ. This means finding all representations (ρ, g) of $D(H)^*$ which solve equation (6.12) in the special case where the flux A is trivial. In this case, we see immediately that all ρ are allowed. The requirement on g is just that $\phi(gx) = \phi(x)$, or equivalently, $\phi(g^{-1}x) = \phi(x)$, for all $x \in H$. Using the invariance property of ϕ, this reduces to $\alpha(x^{-1}gx)\phi(x) = \phi(x)$ and using the invariance property once again, we see that this reduces to the single requirement

$$\alpha(g)\phi(e) = \phi(e)\,. \tag{9.1}$$

Thus, if we define $v := \phi(e) \in V_\alpha$, then g has to be an element of the stabilizer N_v of v. Since ρ was unrestricted, it follows that the residual symmetry algebra is the Hopf subalgebra $\mathcal{T}_v(H)$ of the double which is $F(H) \otimes \mathbb{C}N_v$ as a vector space, or in terms of functions on $H \times H$:

$$\mathcal{T}_v(H) := \{F \in D(H)|\mathrm{supp}(F) \subset H \times N_v\}\,. \tag{9.2}$$

$\mathcal{T}_v(H)$ is a transformation group algebra, with N_v acting on H by conjugation. Hence we may immediately write down all its irreducible representations, using theorem 1. They are labeled by an N_v-orbit $\mathcal{O}$ in H and by a representation τ of the stabilizer $N_\mathcal{O}$ of of a chosen element $g_\mathcal{O} \in \mathcal{O}$ in N_A. We will denote them $\Omega^\mathcal{O}_\tau$ The Hilbert space on which $\Omega^\mathcal{O}_\tau$ acts is the space $F_\tau(N_v, V_\tau)$ defined in (4.7). We will call it $V_\tau^{\mathcal{O}}$ for short. The action of $\Omega_\tau^{\mathcal{O}}$ on this space is given by the formula in theorem 1, which in this case becomes

$$\left(\Omega_\tau^{\mathcal{O}}(F)\phi\right)(x) := \int_{N_v} dz\, F(xg_{\mathcal{O}}x^{-1}, z)\, \phi(z^{-1}x)\,. \tag{9.3}$$

The characters $\psi_\tau^{\mathcal{O}}$ of these representations are given by formula 4.14 or equivalently by formula (4.15). We have

$$\psi_\tau^{\mathcal{O}}(\eta, h) = 1_{N_\eta}(h) 1_{\mathcal{O}}(\eta) \psi_\tau(x_\eta^{-1} h x_\eta)\,. \tag{9.4}$$

Using these characters and the inner product (4.16), one may calculate the fusion rules for $\mathcal{T}_v(H)$-irreps.

Clearly, any representation of $D(H)$ also gives a representation of $\mathcal{T}_v(H)$ by restriction. When we consider the irreps of $D(H)$ as $\mathcal{T}_v(H)$-representations in this way, they will usually no longer be irreducible. Their decomposition into $\mathcal{T}_v(H)$-irreps may be calculated by taking the inner product (4.16) of their character with the characters $\psi_\tau^{\mathcal{O}}$. The character of the $D(H)$-irrep Π_β^B, seen as a $\mathcal{T}_v(H)$-irrep, is just the restriction of the original character χ_β^B; we have

$$\chi_\beta^B(\eta, h) = 1_{N_B}(h) 1_B(\eta) \chi_\alpha(x_\eta^{-1} h x_\eta)\,. \tag{9.5}$$

From this formula and the formula for $\psi_\tau^{\mathcal{O}}$, we see immediately that the irreps $\Omega_\tau^{\mathcal{O}}$ of $\mathcal{T}_v$ which constitute Π_β^B will all have $\mathcal{O} \subset B$. Also we see that a purely magnetic $D(H)$-irrep Π_1^B will decompose into the purely magnetic $\mathcal{T}_v(H)$-irreps $\Omega_1^{\mathcal{O}}$ with $\mathcal{O} \subset B$. A purely electric irrep Π_β^e of $D(H)$ will decompose into the purely electric irreps Ω_τ^e of $\mathcal{T}_v(H)$ which are such, that the irrep τ of $N_A \subset H$ is contained in the decomposition of the irrep β of H.

9.2 Confinement

Let us now determine which representations of the residual algebra $\mathcal{T}_v(H)$ of the previous section will be confined and which will not. The non-confined representations have to satisfy the conditions (7.7). Since $\mathcal{T}_v(H)$ is isomorphic to a transformation group algebra, we may apply the results of section 7.3 (with $N = N_v$ and $K = \{e\}$) to simplify these to the conditions (7.24) and (7.25). From section 7.3, proposition 9, we know that these equations are solved at least by those $\Omega_\tau^{\mathcal{O}}$ for which $g_{\mathcal{O}}K = nK$ for some $n \in N_v$ and τ is trivial on K. Since we have $K = \{e\}$ here, this reduces to just the requirement that $g_{\mathcal{O}} \in N_v$. We have also shown that this set of irreps closes under conjugation and tensor products and that they are in fact the irreps of a quotient $\mathcal{U}_v(H)$ of $\mathcal{T}_v(H)$ that is isomorphic to $D(N_v/(N_v \cap K))$, which is here just $D(N_v)$.

It turns out that the irreps $\Omega_\tau^{\mathcal{O}}$ with $g_{\mathcal{O}} \in N_v$ are actually all the irreps that meet the requirements (7.24) and (7.25). Let us check this. In the case at hand, (7.25) is always satisfied, since g_A is the unit element of H. Thus, we are left with condition (7.24). Since the $K = \{e\}$, this reduces to

$$\phi(x_\eta g_{\mathcal{O}}^{-1} x_\eta^{-1} x) = \phi(x)\,. \tag{9.6}$$

Using the invariance property of ϕ, this becomes $\alpha(x^{-1} x_\eta g_{\mathcal{O}} x_\eta^{-1} x)\phi(x) = \phi(x)$. Multiplying with $\alpha(x)$ from the left and using the invariance of ϕ once more, we see $\alpha(x_\eta g_{\mathcal{O}} x_\eta^{-1})\phi(e)$

must equal $\phi(e)$. If we now recall that $v = \phi(e)$ and that the x_η are elements of N_v, then we see that we are left with the requirement that $g_{\mathcal{O}}$ should be an element of N_v. Thus, the class of solutions that we had already is indeed complete and the non-confined algebra is just the quantum double of N_v.

The fact that the non-confined algebra $\mathcal{U}$ is the quantum double $D(N_v)$ of the stabilizer N_v of the condensate vector comes as no surprise; the original $D(H)$-theory was obtained from a gauge theory with a continuous gauge group G by breaking this group down to H through condensation of an electric excitation. All we have done by condensing one of the electric particles of the $D(H)$-theory is to modify the electric condensate of the G-theory in such a way that the residual gauge group is now N_v rather than H. We referred to this replacement of H with N_v already at the end of section 2 and it is encouraging to see that our formalism for symmetry breaking and confinement in quantum groups produces the result we anticipated there.

The result that the non confined irreps of $\Omega^{\mathcal{O}}_\tau$ of $\mathcal{T}_v$ are exactly those for which $g_{\mathcal{O}} \in N_v$ is also in accordance with our intuitive treatment in section 2; the $\Omega^{\mathcal{O}}_\tau$ whose "flux" $g_{\mathcal{O}}$ acts trivially on the condensate are not confined, because they will have trivial braiding with the condensate. The remaining $\Omega^{\mathcal{O}}_\tau$ will be confined, because they pull strings in the condensate. In fact, all the expectations we voiced in section 2 come true and are now under precise mathematical control. "Hadronic" excitations with overall flux in N_v can be classified by means of the fusion rules of $\mathcal{T}_v$, which can be obtained using the inner product (4.16) on the space of characters. Also, the theory of section 7.2 implies that the classification of strings or domain walls does indeed involve the elements of H/N_v, as we now show.

In section 7.2 we asserted that the string associated with an irrep $\Omega^{\mathcal{O}}_\tau$ may be characterized by the restriction of $\Omega^{\mathcal{O}}_\tau$ to the left or right Hopf kernel of the projection Γ of $\mathcal{T}_v(H)$ onto $D(N_v)$. Let us take the right kernel. From (7.40) we see that the elements of the right kernel are all of the form $f \otimes \delta_e$, where f is constant on left cosets of N_v in H; the right kernel is isomorphic to the algebra of functions on the left N_v-cosets in H The irreps E_{hN} of $\mathrm{RKer}(\Gamma)$ are labeled by these cosets and given by

$$E_{hN}(f \otimes \delta_e) = f(h)\,. \tag{9.7}$$

It is easy to find the restriction of $\Omega^{\mathcal{O}}_\tau$ to $\mathrm{RKer}(\Gamma)$. Let ϕ^i_ζ be the basis elements for $V^{\mathcal{O}}_\tau$ as defined through formula (4.11), that is $\phi^i_\zeta(y) = 1_{x_\zeta N_{\mathcal{O}}}(y)\tau(y^{-1}x_\zeta)e^\tau_i$. Note that the ζ is are in this case just elements of H and that we have $x_\zeta g_{\mathcal{O}} x_\zeta^{-1} = \zeta$. Also, $N_{\mathcal{O}}$ is just the stabilizer of $g_{\mathcal{O}}$ in N. Using this, we have

$$\begin{aligned}(\Omega^{\mathcal{O}}_\tau(f \otimes \delta_e)\phi^i_\zeta)(y) &= f(yg_{\mathcal{O}}y^{-1})\phi(y) \\ &= f(yg_{\mathcal{O}}y^{-1})1_{x_\zeta N_{\mathcal{O}}}(y)\tau(y^{-1}x_\zeta)e^\tau_i \\ &= f(\zeta)\phi(y) = E_{\zeta N}(f)\phi(y)\,.\end{aligned} \tag{9.8}$$

So we see that each of the ϕ^i_ζ spans a one dimensional $\mathrm{RKer}(\Gamma)$-submodule of $\Omega^{\mathcal{O}}_\tau$ isomorphic to the module of $E_{\zeta N}$. This gives the decomposition of $\Omega^{\mathcal{O}}_\tau$ into $\mathrm{RKer}(\Gamma)$ modules: for each ζ in the orbit $\mathcal{O}$, we have d_τ copies of $E_{\zeta N}$. Of course, some of the cosets ζN may coincide and then $E_{\zeta N}$ will occur a multiple of d_τ times in the decomposition. In particular, if $\Omega^{\mathcal{O}}_\tau$ is not confined, then the orbit $\mathcal{O}$ is just a conjugacy class of N and we see that $\Omega^{\mathcal{O}}_\tau$ corresponds to $|\mathcal{O}|d_\tau$ copies of the trivial RKer(Γ)-representation E_N, a result which we showed in general already in section 7.2. Here, it is also easy to see that none of the confined irreps of $\mathcal{T}_v$ has this property. In other words, none of the non-confined irreps pull strings, while all the confined ones do. The result we have got for the labeling of the walls is what we should have expected; a string is created by inserting a flux $g \notin N_v$ into the condensate. This string may be characterized by the fact that, if the condensate state on one side of the string is given by $\phi(0) = v$, then it must be given by $\alpha(g)v$ on the other side. But this means that the fluxes gn, with $n \in N_v$, will all pull the same string as the flux g, since $\alpha(gn)v = \alpha(g)v$. Hence, the string may already be characterized by the coset gN_v. However, the flux g which pulls the string may be transformed into the fluxes ngn^{-1} by gauge transformations with elements $n \in N_v$. Hence the walls should indeed be labeled by the set of cosets $ngn^{-1}N_v$ which is just the set of cosets ζN of the elements ζ in the N_v-orbit of g.

9.3 Examples of electric condensates

9.3.1 Abelian H

Suppose a particle in the irrep Π^e_α of $D(H)$ has condensed in the state $v \in V^e_\alpha$. We have seen that the residual symmetry algebra $\mathcal{T}_v(H)$ is the Hopf subalgebra of $D(H) \cong \mathbb{C}(H \times H) \cong F(H \times H)$ which consists of the functions supported by $H \times N_v$ (cf. 9.2). Because H is abelian, the irrep α is one-dimensional and hence N_v is just the kernel N_α of α. Thus, we have $\mathcal{T}_v(H) \cong F(H \times N_\alpha) \cong \mathbb{C}(H \times N_\alpha)$. Here, the action of N_α on H is trivial, since H is abelian and hence the irreps Ω^h_β of $\mathcal{T}_v$ are labeled by an element h of H and an irrep β of N_α. The decomposition of $D(H)$-irreps into $\mathcal{T}_v(H)$-irreps is straightforward: we have $\Pi^h_\beta \equiv \Omega^h_{\tilde{\beta}}$, where $\tilde{\beta}$ is the restriction of β to N_α.

The irreps Ω^h_β of $\mathcal{T}_v$ which are not confined are those for which $h \in N_v$ and they are of course in one to one correspondence with the irreps of $D(N_v)$. The corresponding Hopf projection $\Gamma : \mathcal{T}_v(H) \to D(N_v)$ is just restriction of the functions in $\mathcal{T}_v(H)$ to N_v in the left argument. The left and right Hopf kernels of Γ coincide and they are both isomorphic to the space of functions on the quotient H/N_v. the representations of this function space are just the evaluation functionals on the classes hN_v and as before, we denote them E_{hN_v}. The restriction of an irrep Ω^h_β of $\mathcal{T}$ to LKer(Γ) is simply given by $\Omega^h_\beta \equiv E_{hN_v}$.

To illustrate what happens a bit more explicitly, we will work out the case of $H = \mathbb{Z}_n$. This group is generated by a single element which we will call r and it has n irreps $\alpha_0, \ldots, \alpha_{n-1}$, given by

$$\alpha_k : \; r^a \mapsto e^{2\pi i k a/n} \,. \tag{9.9}$$

The kernel of α_k consists of those r^a for which $ak = 0 \bmod n$. The minimal non-zero a for which this holds is the quotient $n/\mathrm{gcd}(n,k) =: x$. Hence, we have $N_\alpha = \langle\, r^x \,\rangle \cong \mathbb{Z}_{\mathrm{gcd}(n,k)}$. The corresponding residual symmetry algebra is

$$\mathcal{T}_v(\mathbb{Z}_n) \cong F(\mathbb{Z}_n \times \mathbb{Z}_{\mathrm{gcd}(n,k)}) \cong \mathbb{C}\mathbb{Z}_n \times \mathbb{Z}_{\mathrm{gcd}(n,k)} \,. \tag{9.10}$$

Thus we see that the electric symmetry can be broken in as many different ways as n has divisors (the magnetic symmetry is never broken). The irreps $\Omega^{r^a}_{\alpha_l}$ of the residual algebra are labeled by an element $r^a \in H$ and an irrep α_l (with $0 \leq l < \gcd(n,k)$) of $\mathbb{Z}_{\gcd(n,k)}$. The decomposition of the irreps $\Pi^{r^a}_{\alpha_l}$ (with $0 \leq l < n$) of $\mathbb{Z}_n$ is then given by

$$\Pi^{r^a}_{\alpha_l} \equiv \Omega^{r^a}_{\alpha_{l \bmod \gcd(n,k)}} . \tag{9.11}$$

This follows from the fact that $\alpha_l(r^{px}) = e^{2\pi i l p x/n} = e^{2\pi i l p/\gcd(n,k)}$.

The irreps $\Omega^{r^a}_{\alpha_l}$ which are not confined are those for which $r^a \in N_\alpha$, i.e. $a = px$, with $0 \leq p < \gcd(n,k)$. These correspond to the irreps $\Pi^{p}_{\alpha_l}$ of the non-confined algebra $\mathcal{U}$, which is given by

$$\mathcal{U}_v(\mathbb{Z}_n) \cong D(N_\alpha) \cong D(\mathbb{Z}_{\gcd(n,k)}) . \tag{9.12}$$

The Hopf map Γ from $F(\mathbb{Z}_n \times \mathbb{Z}_{\gcd(n,k)})$ to $\mathcal{U}$ is just restriction to $\mathbb{Z}_{\gcd(n,k)}$ in the left argument. The Hopf kernel of this map consist of the functions $f \otimes \delta_e$ in $F(\mathbb{Z}_n \times \mathbb{Z}_{\gcd(n,k)})$ for which f is constant on the cosets of $N_\alpha = \mathbb{Z}_{\gcd(n,k)}$. Hence, we have

$$\mathrm{LKer}(\Gamma) \cong F(\mathbb{Z}_x) . \tag{9.13}$$

The representations of $\mathrm{LKer}(\Gamma)$ may be denoted E_a with $(0 \leq a < x)$ and are given by $E_a(f \otimes \delta_e) = f(r^a)$. The restriction of $\Omega^{r^a}_{\alpha_l}$ to $\mathrm{LKer}(\Gamma)$ is given by

$$\Omega^{r^a}_{\alpha_l} \equiv E_{a \bmod x} . \tag{9.14}$$

9.3.2 $H = D_{2m+1}$

We will treat all the possible types of electric condensate in order.

1. First, we take a condensate $v \in V^e_{J_1}$. We then have $N_v = \langle\, r \,\rangle \cong \mathbb{Z}_{2m+1}$ and hence

$$\mathcal{T}_v(D_{2m+1}) \cong F(D_{2m+1}) \,\tilde{\otimes}\, \mathbb{C}\mathbb{Z}_{2m+1} . \tag{9.15}$$

   Here and in the sequel, the tilde on the tensor product sign indicates that the factor on the right acts on the factor on the left through conjugation. To find the irreps of $\mathcal{T}_v$, we first need to find the orbits of the adjoint $\mathbb{Z}_{2m+1}$-action on D_{2m+1} and their stabilizers. One easily finds that the orbits are $\{e\}, \{r\}, \{r^2\}, \ldots, \{r^{2m}\}$ and $\{s, sr, sr^2, \ldots, sr^{2m}\}$ Of these, all the orbits that contain a single element have stabilizer $\langle\, r \,\rangle \cong \mathbb{Z}_{2m+1}$, while the remaining orbit has the trivial stabilizer $\{e\}$. Thus, the irreps of $\mathcal{T}_v$ may be denoted $\Omega^{r^k}_{\beta_l}$ (with $0 \leq k, l < 2m+1$) and Ω^s. Here, we let r^k and s denote the orbits of r^k and s, in order not to overload the notation. We see that $\mathcal{T}_v$ has $(2m+1)^2+1$ irreps, which are all one-dimensional, except for Ω^s, which is $2m+1$-dimensional. It follows that the squares of the dimensions add up to $2(2m+1)^2$, which equals the dimension of $\mathcal{T}_v$, as it should. The decomposition of $D(D_{2m+1})$-irreps into $\mathcal{T}_v$-irreps may be found directly or by means of the orthogonality relations for the characters of $\mathcal{T}_v$. We have

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega^e_{\beta_0} & \Pi^{r^k}_{\beta_l} \equiv \Omega^{r^k}_{\beta_l} \oplus \Omega^{r^{-k}}_{\beta_{-l}} \\ \Pi^e_{J_1} \equiv \Omega^e_{\beta_0} & \Pi^s_{\gamma_0} \equiv \Omega^s \\ \Pi^e_{\alpha_k} \equiv \Omega^e_{\beta_k} \oplus \Omega^e_{\beta_{-k}} & \Pi^s_{\gamma_1} \equiv \Omega^s . \end{array} \tag{9.16}$$

Of the representations of $\mathcal{T}_v$, Ω^s is confined, since $s \notin N_v$. The others are not confined and are in one to one correspondence with the irreps of the non-confined algebra

$$\mathcal{U}_v(D_{2m+1}) \cong D(N_v) = D(\mathbb{Z}_{2m+1})\,. \tag{9.17}$$

The right Hopf kernel of the projection Γ of $\mathcal{T}_v(H)$ onto $D(N_v)$ is isomorphic to the algebra of functions on the set of left $\langle\, r\,\rangle$-cosets. There are only two such cosets, namely $R := \langle\, r\,\rangle$ and $S := s\langle\, r\,\rangle$ and hence two corresponding one-dimensional representations E_R and E_S of the right kernel. The decomposition of $\mathcal{T}_v$-irreps into RKer(Γ)-irreps is given by

$$\Omega^{r^k}_{\beta_l} \equiv E_R \Omega^s \equiv (2m+1)E_S\,. \tag{9.18}$$

2. Now we take a condensate v in the module $V^e_{\alpha_j}$. The stabilizer N_v of v consists by definition of all the elements g of $D(D_{2m+1})$ for which v is an eigenvector of $\alpha_j(g)$ with eigenvalue 1. This includes in particular all the elements of the kernel of α_j. From the character table of $D(D_{2m+1})$ (table 1), one may read off that this kernel consists of those elements r^a for which $q^{ja} + q^{-ja} = 2$, where $q = e^{2\pi i/(2m+1)}$, or in other words, for which $\cos(2\pi ja/(2m+1)) = 1$. It follows that one has to have $ja = 0 \bmod 2m+1$. The smallest non-zero a for which this holds is $(2m+1)/\gcd(2m+1,j) =: x$. Thus, one has $N_{\alpha_j} = \langle\, r^x\,\rangle \cong \mathbb{Z}_{\gcd(2m+1,j)}$.

   Of course, the stabilizer N_v of v may be larger than N_{α_j}, if v is an eigenvector of $\alpha_j(g)$ for some $g \notin N_{\alpha_j}$. Thus, in order to find out what kinds of stabilizers are possible, it is a good idea to have a look at the eigenvalues of the matrices $\alpha_j(g)$. From the explicit matrices in (5.7), we see that the eigenvalues of $\alpha_j(r^p)$ are q^{jp} and q^{-jp}, with $q = e^{2\pi i/(2m+1)}$. It follows that, if one of the eigenvalues of r^p equals 1, so does the other. Hence, the only elements of $\langle\, r\,\rangle$ whose matrices have eigenvalues equal to one are those that are already contained in the kernel of α_j. The eigenvalues of each of the matrices $\alpha_j(sr^p)$ are 1 and -1. Thus, we have two possibilities: either v is not left invariant by any of the matrices $\alpha_j(sr^p)$, in which case $N_v = N_{\alpha_j} = \langle\, r^x\,\rangle \cong \mathbb{Z}_{\gcd(2m+1,j)}$, or v is left invariant by some of the $\alpha_j(sr^p)$. In this case, we may without loss of generality choose v to be the invariant vector of $\alpha_j(s)$, since each of the sr^p is a conjugate of s in D_{2m+1} and hence the invariant vectors of the sr^p are in the same gauge orbit as the invariant vector of s. With this choice, one sees easily that $N_v = \langle\, r^x\,\rangle \cup s\langle\, r^x\,\rangle \cong D_{\gcd(2m+1,j)}$. We will now treat the two possibilities for N_v in order.

2.a When $N_v = \langle\, r^x\,\rangle \cong \mathbb{Z}_{\gcd(2m+1,j)}$, we have

$$\mathcal{T}_v(D_{2m+1}) \cong F(D_{2m+1}) \,\tilde{\otimes}\, \mathbb{C}\mathbb{Z}_{\gcd(2m+1,j)}\,. \tag{9.19}$$

The orbits of the $\langle\, r^x\,\rangle$-action on D_{2m+1} are $\{e\}, \{r\}, \ldots, \{r^{2m}\}$, with stabiliser $\langle\, r^x\,\rangle$, and $s\langle\, r^x\,\rangle$, $sr\langle\, r^x\,\rangle, \ldots, sr^{x-1}\langle\, r^x\,\rangle$, with stabilizer $\{e\}$. This means that the irreps of $\mathcal{T}$ may be denoted as $\Omega^{r^k}_{\beta_l}$ (with $0 \le k < 2m+1$, $0 \le l < \gcd(2m+1,j)$) and

Ω^{sr^k} (with $0 \leq k < x$). Here, we have once again denoted orbits by representative elements. We see that there are $(2m+1)\mathrm{gcd}(2m+1,j) + (2m+1)/\mathrm{gcd}(2m+1,j)$ irreps. Of these, $(2m+1)\mathrm{gcd}(2m+1,j)$ are one-dimensional and the remaining ones (the Ω^{sr^p}) are $\mathrm{gcd}(2m+1,j)$-dimensional, so that the squares of the dimensions again add up to the dimension of $\mathcal{T}$, which is $2(2m+1)\mathrm{gcd}(2m+1,j)$. The decomposition of $D(D_{2m+1})$-irreps reads

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega^e_{\beta_0} & \Pi^{r^k}_{\beta_l} \equiv \Omega^{r^k}_{\beta_l} \oplus \Omega^{r^{-k}}_{\beta_{-l}} \\ \Pi^e_{J_1} \equiv \Omega^e_{\beta_0} & \Pi^s_{\gamma_0} \equiv \bigoplus_{0\leq p<x} \Omega^{sr^p} \\ \Pi^e_{\alpha_l} \equiv \Omega^e_{\beta_l} \oplus \Omega^e_{\beta_{-l}} & \Pi^s_{\gamma_1} \equiv \bigoplus_{0\leq p<x} \Omega^{sr^p} \end{array} \tag{9.20}$$

where the labels l and $-l$ should be read modulo $2m+1$ on the left hand side and modulo $\mathrm{gcd}(2m+1,j)$ on the right hand side. The non-confined irreps are those $\Omega^{r^k}_{\beta_l}$ for which $r^k \in \langle\, r^x \,\rangle$ and they are in one correspondence with the irreps of the non-confined algebra

$$\mathcal{U}_v(D_{2m+1}) \cong D(\mathbb{Z}_{\mathrm{gcd}(j,2m+1)})\,. \tag{9.21}$$

The right and the left kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ are equal and isomorphic to the algebra of functions on the quotient group $D_{2m+1}/\langle\, r^x \,\rangle$. Since this quotient group is isomorphic to D_x, we have

$$\mathrm{RKer}(\Gamma) \cong F(D_x)\,. \tag{9.22}$$

The representations of $\mathrm{RKer}(\Gamma)$ are labeled by the elements R^k, SR^k of D_x and we denote them E_{R^k}, E_{SR^k}. The decomposition of $\mathcal{T}$-irreps into $\mathrm{RKer}(\Gamma)$-irreps is given by

$$\Omega^{r^k}_{\beta_l} \equiv E_{R^k} \qquad \Omega^{sr^k} \equiv xE_{SR^k} \tag{9.23}$$

where, on the right hand side, k should be taken modulo x.

2.b When $N_v \cong D_{\mathrm{gcd}(2m+1,j)}$, we have

$$\mathcal{T}_v(D_{2m+1}) \cong F(D_{2m+1}) \tilde{\otimes} \mathbb{C}D_{\mathrm{gcd}(2m+1,j)}\,. \tag{9.24}$$

The orbits of the $D_{\mathrm{gcd}(2m+1,j)}$-action on D_{2m+1} are $\{e\}$, $\{r, r^{-1}\}$, $\{r^2, r^{-2}\}, \ldots,$ $\{r^m, r^{-m}\}$, $s\langle\, x \,\rangle$ and $sr\langle\, x \,\rangle \cup sr^{x-1}\langle\, x \,\rangle$, $sr^2\langle\, x \,\rangle \cup sr^{x-2}\langle\, x \,\rangle \ldots$. The stabilizer of e is of course all of $D_{\mathrm{gcd}(2m+1,j)}$, the stabilizer of r^k is $\langle\, r^x \,\rangle \cong \mathbb{Z}_{\mathrm{gcd}(2m+1,j)}$ and the stabilizer of s is $\langle\, s \,\rangle \cong \mathbb{Z}_2$. The stabilizer of the orbits $sr^p\langle\, x \,\rangle \cup sr^{x-p}\langle\, x \,\rangle$ is just $\{e\}$. Hence, the irreps of $\mathcal{T}_v$ may be denoted $\Omega^e_{J_0}$, $\Omega^e_{J_1}$, $\Omega^e_{\alpha_k}$ (with $1 \leq k \leq \frac{1}{2}(\mathrm{gcd}(2m+1,j)-1)$), $\Omega^{r^k}_{\beta_l}$ (with $0 < k \leq m$, $0 \leq l < \mathrm{gcd}(2m+1,j)$), $\Omega^s_{\gamma_0}, \Omega^s_{\gamma_1}$ and finally Ω^{sr^p} (with $1 \leq p < \frac{1}{2}(x-1)$). This yields $3 + \frac{1}{2}(2m+1)(\mathrm{gcd}(2m+1,j) + 1/\mathrm{gcd}(2m+1,j))$ irreps in total and one may check that the squares of their dimensions sum correctly to the dimension of $\mathcal{T}_v$, which is $4(2m+1)\mathrm{gcd}(2m+1,j)$. The decomposition of the $\Pi^e_{\alpha_k}$ into $\mathcal{T}_v$-irreps is now

$$\Pi^e_{\alpha_k} \equiv \begin{cases} \Omega^e_{J_0} \oplus \Omega^e_{J_1} & ([k]=0) \\ \Omega^e_{\alpha_{[k]}} & ([k] \leq \frac{1}{2}(\mathrm{gcd}(2m+1,j)-1) \\ \Omega^e_{\alpha_{\mathrm{gcd}(2m+1,j)-[k]}} & ([k] > \frac{1}{2}(\mathrm{gcd}(2m+1,j)-1)\,. \end{cases} \tag{9.25}$$

Here, $[k]$ denotes k mod $\gcd(2m+1,j)$. The decomposition of the other D_{2m+1}-irreps into $\mathcal{T}_v$-irreps is given by

$$\begin{aligned}
\Pi^e_{J_0} &\equiv \Omega^e_{J_0} \quad \Pi^s_{\gamma_0} \equiv \Omega^{s\gamma_0} \oplus \bigoplus_{1\le p<\frac{1}{2}(x-1)} \Omega^{sr^p} \\
\Pi^e_{J_1} &\equiv \Omega^e_{J_1} \quad \Pi^s_{\gamma_1} \equiv \Omega^{s\gamma_1} \oplus \bigoplus_{1\le p<\frac{1}{2}(x-1)} \Omega^{sr^p} \\
2\Pi^{r^k}_{\beta_l} &\equiv \Omega^{r^k}_{\beta_l}
\end{aligned} \tag{9.26}$$

The labels l on the left should be read modulo $2m+1$, while those on the right hand side should be read modulo $\gcd(2m+1,j)$. The non-confined irreps of $\mathcal{T}_v$ are $\Omega^e_{J_0}, \Omega^e_{J_1}$, the $\Omega^e_{\alpha_k}$, $\Omega^s_{\gamma_0}, \Omega^s_{\gamma_1}$ and those $\Omega^{r^k}_{\beta_l}$ for which $r^k \in \langle r^x \rangle$. These irreps correspond to the irreps of the non-confined algebra

$$\mathcal{U}_v(D_{2m+1}) \cong D\left(D_{\gcd(2m+1,j)}\right). \tag{9.27}$$

The right kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ is isomorphic to the algebra of functions on the space of left cosets of $N_v \cong D_{\gcd(2m+1,j)}$ in D_{2m+1}. There are x distinct cosets, namely the cosets of $e, r, \ldots, r^{x-1}$. We will denote these $E, R, \ldots, R^{x-1}$. The corresponding irreps of $\mathrm{RKer}(\Gamma)$ will again be denoted E_{R^k}. The restriction of the irreps of $\mathcal{T}_v$ to $\mathrm{LKer}(\Gamma)$ is given by

$$\begin{array}{lll}
\Omega^e_{J_0} \equiv E_E & \Omega^{r^k}_{\beta_l} \equiv E_{R^k} & \Omega^{sr^p} \equiv \gcd(2m+1,j)(E_{R^p} \oplus E_{R^{x-p}}) \\
\Omega^e_{J_1} \equiv E_E & \Omega^{s\gamma_0} \equiv \gcd(2m+1,j)E_E & \\
\Omega^e_{\alpha_l} \equiv 2E_E & \Omega^{s\gamma_1} \equiv \gcd(2m+1,j)E_E &
\end{array} \tag{9.28}$$

where the index k should be read modulo $2m+1$ on the left hand side and modulo x on the right. In the restriction of Ω^{sr^p}, we see our first example of a situation where the wall created by a $\mathcal{T}$-particle carries a representation of $\mathrm{RKer}(\Gamma)$ that contains two distinct irreps of $\mathrm{RKer}(\Gamma)$, namely E_{R^p} and $E_{R^{x-p}}$. The isotypical components of these irreps are sent onto each other by gauge transformation with $s \in N_v$, since $sR^ps^{-1} = R^{x-p}$.

10. Gauge invariant magnetic condensates

10.1 Symmetry breaking

There is precisely one gauge invariant state in every magnetic representation Π^A_1. This state is represented by the constant function

$$\phi : h \mapsto 1 \tag{10.1}$$

on H. To find the Hopf stabilizer of ϕ, we need to find the irreps (ρ, g) of $D(H)^*$ which solve equation (6.12). Since ϕ is constant equal to one, this reduces to

$$\rho(g_A) = I\,. \tag{10.2}$$

Hence, the unbroken symmetry algebra is the algebra generated by the matrix elements of the representations (ρ, g) for which g_A is contained in the kernel of ρ. Now define K_A as the minimal normal subgroup of H that contains g_A (and hence all of A). Since the kernel of a representation is a normal subgroup, the irreducible representations ρ which have g_A in their kernel will be precisely the ones which contain all of K_A in their kernel. Such irreps are in one-to-one correspondence with the irreps of H/K_A [38] and since the matrix elements of the irreps of H/K_A generate $F(H/K_A)$, the algebra generated by the matrix elements of the irreps of H which contain g_A in their kernel is precisely the algebra $\mathcal{T}_A$ of functions on G which are constant on the cosets of K_A. Hence, the unbroken symmetry algebra in this case is the Hopf subalgebra $\mathcal{T}_A(H)$ of $D(H)$ defined by

$$\mathcal{T}_A(H) := \{F \in D(H) | \forall k \in K_A : F(xk, y) = F(x, y)\} . \tag{10.3}$$

Clearly, $\mathcal{T}_A \cong F(H/K_A \times H)$ as a linear space and we see that $\mathcal{T}_A$ is a transformation group algebra, with H acting on K_A by conjugation. This means we can once again make use of theorem 1 to write down the irreps of $\mathcal{T}_A$. They are labeled by an H-orbit $\mathcal{O} \subset H/K$ and an irrep τ of the stabilizer $N_{\mathcal{O}}$ of a chosen element $g_{\mathcal{O}} \in \mathcal{O}$. The irrep labeled by $\mathcal{O}$ and τ will be denoted $\Omega_\tau^{\mathcal{O}}$. It acts on the Hilbert space $F_\tau(H, V_\tau)$ in the usual way:

$$\left(\Omega_\tau^{\mathcal{O}}(F)\phi\right)(x) := \int_H dz\, F(x g_{\mathcal{O}} x^{-1}, z)\, \phi(z^{-1}x) . \tag{10.4}$$

The character $\psi_\tau^{\mathcal{O}}$ of $\Omega_\tau^{\mathcal{O}}$ is given as a function on $H/K_A \times H$ by (cf. (4.15))

$$\psi_\tau^{\mathcal{O}}(\eta, h) = 1_{N_\eta}(h) 1_{\mathcal{O}}(\eta) \psi_\tau(x_\eta^{-1} h x_\eta) . \tag{10.5}$$

The decomposition of any $\mathcal{T}_A(H)$-module into irreps may be found by calculating the inner products (defined in (4.16)) between the character of the module and the above characters of the irreps. Of course, we can view any $D(H)$-module as a $\mathcal{T}_A(H)$-module by restriction. The characters χ_β^B of the irreps Π_β^B of $D(H)$, viewed as $\mathcal{T}_A(H)$-modules are given by

$$\chi_\beta^B(gK_A, h) = \sum_{k \in K_A} 1_{N_{gk}}(h) 1_B(gk) \chi_\beta(x_{gk}^{-1} h x_{gk}) . \tag{10.6}$$

We see that all the irreps $\Omega_\tau^{\mathcal{O}}$ in the decomposition of Π_β^B must be such that B is a subset of the set of elements of H that constitute the K_A-classes in $\mathcal{O}$. Clearly, there is only a single orbit $\mathcal{O}$ for which this holds. The decomposition of a purely electric representation Π_β^e is very simple: such a representation is irreducible and isomorphic to the purely electric irrep $\Omega_\beta^{\mathcal{O}=K_A}$ (Note that $N_{K_A} = H$). On the other hand, the decomposition of a purely magnetic representation Π_1^B may contain irreps $\Omega_\tau^{\mathcal{O}}$ which are not purely magnetic (i.e. τ may be non-trivial).

10.2 Confinement

We will now find out which of the irreps $\Omega_\tau^{\mathcal{O}}$ of $\mathcal{T}_A$ are confined and which are not. The non-confined irreps have to satisfy the requirements (7.7). Since $\mathcal{T}_A$ is isomorphic to a transformation group algebra, these reduce to the conditions (7.24) and (7.25), with $K =$

K_A and $N = H$. We have seen in section 7.3, proposition 9, that these requirements will be satisfied by the set of irreps $\Omega^{\mathcal{O}}_{\tau}$ for which $g_{\mathcal{O}}K = nK$ for some $n \in N$ and for which τ is trivial on K. The first of these requirements is trivial here, since $N = H$ and so this set consist of all $\Omega^{\mathcal{O}}_{\tau}$ for which τ is trivial on K_A. These irreps correspond to the irreps of the quotient $D(H/K_A)$ of $\mathcal{T}_A$. In the case at hand, it turns out that this set of of solutions is actually complete and hence the non-confined symmetry algebra $\mathcal{U}_A$ is just the quantum double of the quotient group H/K_A. Let us demonstrate this.

Equation (7.24) is trivially satisfied by the matrix elements of all the irreps of $\mathcal{T}_A$, since ϕ is the constant function 1. Thus, we are left with the requirement (7.25). Since the support of ϕ is all of H, this becomes

$$(\forall x \in H) \qquad \tau(x g_A x^{-1}) = I \tag{10.7}$$

or in other words

$$A \subset \mathrm{Ker}(\tau)\,. \tag{10.8}$$

The requirement that $A \subset \mathrm{Ker}(\tau)$ is equivalent to the requirement that $K_A \subset \mathrm{Ker}(\tau)$, since K_A is just the subgroup of H generated by the elements of A. Hence, the non-confined irreps $\Omega^{\mathcal{O}}_{\tau}$ of $\mathcal{T}_A$ are exactly those for which τ is trivial on K_A, as we claimed.

The results we have obtained are quite satisfying when one thinks back of the intuition that went into our method of finding the non-confined irreps. We wanted the non-confined irreps to have trivial braiding with the condensate. For a purely magnetic condensate, this means roughly that the flux of the condensate should commute with the flux of the non-confined irreps and should act trivially on the charges of the non-confined irreps. The first of these conditions is automatically met: the flux state of the condensate commutes with any other flux state (the class sum is a central element of the group algebra of H). Therefore, there is no requirement on $\mathcal{O}$. The second condition is implemented by the demand that τ is trivial on K_A, the group which is generated by the fluxes in the class A. We also wanted to have well-defined fusion, spin and braiding among the non-confined particles and these are now provided by the Hopf structure, R-matrix and ribbon-element of $D(H/K_A)$.

Finally, let us say something about the characterization of strings (or walls). From proposition 10, we see that the Hopf kernel of the projection $\Gamma : \mathcal{T}_A(H) \to D(H/K_A)$ is just the set of elements $1 \otimes f \in \mathcal{T}_A(H)$ for which f has support in K_A. In this case, the left and right Hopf kernels coincide and hence the kernel is itself a Hopf algebra. This Hopf algebra is clearly isomorphic to the group algebra $\mathbb{C}K_A$ (cf. corrollary 3) and hence the irreps of $\mathrm{LKer}(\Gamma)$ correspond to the irreps of K_A. If ρ is an irrep of K_A then we also write ρ for the corresponding irrep of $\mathrm{LKer}(\Gamma)$ and with this slight abuse of notation, we may write

$$\rho(1 \otimes \delta_k) = \rho(k) \tag{10.9}$$

for all $k \in K_A$. We will now calculate the decomposition of a representation $\Omega^{\mathcal{O}}_{\tau}$ of $\mathcal{T}_A(H)$ into representations of $\mathrm{LKer}(\Gamma)$ by means of the formula (4.13) for the character $\psi^{\mathcal{O}}_{\tau}$ of $\Omega^{\mathcal{O}}_{\tau}$.

For $g \in K_A$, we have

$$(\psi_\tau^{\mathcal{O}}(1 \otimes \delta_g) = \int_{\mathcal{O}} d\zeta \int_{N_{\mathcal{O}}} dn\, \delta_g(x_\zeta n x_\zeta^{-1})\chi_\tau(n)$$
$$= \int_{\mathcal{O}} d\zeta\, \chi_\tau(x_\zeta^{-1} g x_\zeta)\,. \tag{10.10}$$

From this, we see that the restriction of $\Omega_\tau^{\mathcal{O}}$ to $\mathrm{LKer}(\Gamma) \cong \mathbb{C}K_A$ contains exactly the irreps of K_A that are contained in the restriction of β to K_A, together with the irreps obtained from these by composition with the automorphisms of K_A that are given by conjugation with the x_ζ^{-1}. As in the case of electric condensates, we see that the non-confined irreps are exactly all those that have trivial restriction to the Hopf kernel of Γ.

10.3 Examples of gauge invariant condensates

10.3.1 Abelian H

For abelian H, every state in a purely magnetic representation Π^A is gauge invariant, so this section covers all purely magnetic condensates for abelian groups. Suppose we condense a state in the purely magnetic representation labeled by the element g_A of H. Then we know that the residual symmetry algebra $\mathcal{T}_A(H)$ is the Hopf subalgebra of $D(H)$ which consists of the functions that are constant on cosets of K_A in their left argument. Here K_A is the minimal normal subgroup of H that contains g_A, which, when H is abelian, is just the cyclic group generated by g_A. As an algebra, $\mathcal{T}_A(H)$ is isomorphic to the transformation group algebra $F(H/K_A \times H)$, where H acts on H/K_A by conjugation. When H is abelian, the action of H on H/K_A is thus trivial. The orbits are then just the elements of $H/K_A = H/\langle\, g_A \,\rangle$ and the stabilizer of each orbit is all of H. Thus, the irreps of $\mathcal{T}_A(H)$ may be denoted $\Omega_\alpha^{hK_A}$, where hK_A is an element of H/K_A and α is an irrep of H. The action of $\mathcal{T}_A$ in the irrep $\Omega_\alpha^{hK_A}$ is given in formula (10.4). The irreps of $D(H)$ may be easily decomposed into irreps of $\mathcal{T}_A(H)$; we have $\Pi_\alpha^h \equiv \Omega_\alpha^{hK_A}$. The non-confined irreps of $\mathcal{T}_A(H)$ are those $\Omega_\alpha^{hK_A}$ for which α is trivial on K_A These correspond to the irreps of the non-confined algebra $\mathcal{U} \cong D(H/K_A) \cong D(H/\langle\, g_A \,\rangle)$. The kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ is isomorphic to $\mathbb{C}K_A$ and hence its irreps are just the irreps of K_A. Since $K_A = \langle\, g_A \,\rangle$, it follows that the number of these irreps equals the order of the element g_A. We may indeed give the irreps explicitly; denoting them as ρ_k (with $0 \le k < \mathrm{ord}(g_A)$), we have $\rho_k((g_A)^p) = \exp(2\pi i k p/\mathrm{ord}(g_A))$ The restriction of the irreps of $\mathcal{T}_A$ to $\mathrm{LKer}(\Gamma)$ is also easily found. We have $\Omega_\alpha^{hK_A} \equiv \alpha|_{K_A}$. In other words, the wall corresponding to $\Omega_\alpha^{hK_A}$ can be labeled by the phase factor $\alpha(g_A)$.

We once again explicitly work out the case of $H = \mathbb{Z}_n$. As in section 9.3.1, we will denote our preferred generator for $\mathbb{Z}_n$ by r and we write $\alpha_0, \ldots, \alpha_{n-1}$ for the irreps of $\mathbb{Z}_n$. Now suppose we condense the magnetic flux $g_A = r^k$. Then we have $K_A = \langle\, r^k \,\rangle = \langle\, r^{\mathrm{gcd}(k,n)} \,\rangle \cong \mathbb{Z}_x$, where $x = n/\mathrm{gcd}(k,n)$. As a consequence, we have $H/K_A \cong \mathbb{Z}_n/\mathbb{Z}_x \cong \mathbb{Z}_{\mathrm{gcd}(k,n)}$. Thus,

$$\mathcal{T}_{[r^k]}(\mathbb{Z}_n) \cong F(\mathbb{Z}_{\mathrm{gcd}(k,n)} \times \mathbb{Z}_n) \cong \mathbb{C}(\mathbb{Z}_{\mathrm{gcd}(k,n)} \times \mathbb{Z}_n) \tag{10.11}$$

and we see that there is one type of broken symmetry for each divisor of n. The irreps of $\mathcal{T}$ may be labeled by an element $r^a K_A$ of $\mathbb{Z}_{\mathrm{gcd}}(k,n)$ and an irrep α_l of $\mathbb{Z}_n$ and we will denote them $\Omega^{r^a K_A}_{\alpha_l}$. The decomposition of the irreps $\Pi^{r^a}_{\alpha_l}$ (with $0 \leq a < n$) of $\mathbb{Z}_n$ is then given by

$$\Pi^{r^a}_{\alpha_l} \equiv \Omega^{r^a \bmod \gcd(n,k) K_A}_{\alpha_l} . \tag{10.12}$$

The unconfined irreps of $\mathcal{T}$ are those $\Omega^{r^a K_A}_{\alpha_l}$ for which $\alpha_l(r^k) = \exp(2\pi i k l/n) = 1$, or equivalently $kl = 0 \bmod n$. These are exactly the ones for which l is a multiple of x and we see that they correspond to the irreps of

$$\mathcal{U}_{[r^k]}(\mathbb{Z}_n) \cong D(H/\langle\, r^k\,\rangle) \cong D(\mathbb{Z}_{\gcd(k,n)})\,. \tag{10.13}$$

The kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ is isomorphic to $\mathbb{C}\mathbb{Z}_x$ and has representations $\tilde{\alpha}_j$ (with $(0 \leq j < x)$) defined in the usual way, with $r^{\gcd(k,n)}$ as the preferred generator. That is, we take $\tilde{\alpha}_j(r^{\gcd(k,n)}) = \exp(2\pi i j/x)$. The restriction of $\mathcal{T}$-irreps to $\mathrm{RKer}(\Gamma)$ is given by

$$\Omega^{r^a K_A}_{\alpha_l} \equiv \tilde{\alpha}_{l \bmod x}\,. \tag{10.14}$$

One should notice the duality between the situation described here and that for symmetry breaking by electric condensates described in section 9.3.1.

10.3.2 $H = D_{2m+1}$

- First, we take the condensate state in the module $V^{r^k}_{\beta_0}$. To find the residual symmetry algebra $\mathcal{T}_{[r^k]}$, we first need to find the minimal normal subgroup $K_{[r^k]}$ of D_{2m+1} that contains r^k. This is just the subgroup generated by the elements of the conjugacy class of r^k, which are r^k and r^{-k}. In other words, we have $K_{[r^k]} = \langle\, r^k\,\rangle = \langle\, r^{\gcd(k,2m+1)}\,\rangle \cong \mathbb{Z}_x$, where $x = (2m+1)/\gcd(k,2m+1)$. One checks easily that $D_{2m+1}/\langle\, r^k\,\rangle \cong D_{\gcd(k,2m+1)}$, where this $D_{\gcd(k,2m+1)}$ is generated in the usual way by the rotation $R = r\langle\, r^k\,\rangle$ and the reflection $S = s\langle\, r^k\,\rangle$. We will also use the notation E for the class $e\langle\, r^k\,\rangle$. The residual algebra is now

$$\mathcal{T}_{[r^k]}(D_{2m+1}) \cong F(D_{\gcd(k,2m+1)})\,\tilde{\otimes}\,\mathbb{C}D_{2m+1}\,. \tag{10.15}$$

  The orbits of the D_{2m+1} action are exactly the conjugacy classes of $D_{\gcd(k,2m+1)}$. These are $\{E\}$, $\{R^p, R^{-p}\}$ (with $0 \leq p < (\gcd(k,2m+1)-1)/2$) and $\{S, SR, \ldots, SR^{(\gcd(k,2m+1)-1)}\}$. The stabilizers of these orbits are $N_E = D_{2m+1}$, $N_{R^p} = \langle\, r\,\rangle \cong \mathbb{Z}_{2m+1}$ and $N_S = E \cup sE \cong D_x$. Thus the irreps of $\mathcal{T}_{[r^k]}$ may be written as $\Omega^E_{J_0}$, $\Omega^E_{J_1}$, $\Omega^E_{\alpha_j}$ (with $1 \leq j \leq m$), $\Omega^{R^p}_{\beta_l}$ (with $1 \leq p \leq \frac{1}{2}(\gcd(k,2m+1)-1)$, $0 \leq l < 2m+1$), $\Pi^S_{J_0}$, $\Pi^S_{J_1}$ and $\Pi^S_{\alpha_j}$ (with $1 \leq j \leq \frac{1}{2}(x-1)$). This yields $3 + \frac{1}{2}(2m+1)(\gcd(k,2m+1) + \frac{1}{\gcd(k,2m+1)})$ irreps in total and one may check that the squares of their dimensions add up to the dimension of $\mathcal{T}_{[r^k]}$, which is $4\gcd(k,2m+1)(2m+1)$. The decomposition of $D(D_{2m+1})$-irreps is as follows:

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega^E_{J_0} & \Pi^{r^p}_{\beta_l} \equiv \Omega^{R^p}_{\beta_l} \\ \Pi^e_{J_1} \equiv \Omega^E_{J_1} & \Pi^s_{\gamma_0} \equiv \Omega^S_{J_0} \oplus \bigoplus_j \Omega^S_{\alpha_j} \\ \Pi^e_{\alpha_l} \equiv \Omega^E_{\alpha_l} & \Pi^s_{\gamma_1} \equiv \Omega^S_{J_1} \oplus \bigoplus_j \Omega^S_{\alpha_j}\,. \end{array} \tag{10.16}$$

  In the decomposition of $\Pi^s_{\gamma_0}$, we see that a chargeless flux may be turned into a charged flux upon formation of a magnetic condensate.

The irreps $\Omega^{\mathcal{O}}_{\tau}$ of $\mathcal{T}_{[r^k]}$ that are not confined are exactly those for which τ is trivial on $K_{[r^k]} = \langle r^k \rangle$. These are $\Omega^E_{J_0}, \Omega^E_{J_1}$, the $\Omega^E_{\alpha_l}$ and $\Omega^{Rp}_{\alpha_l}$ for which l is a multiple of x, $\Omega^S_{J_0}$ and $\Omega^S_{J_1}$. These correspond to the irreps of the non-confined algebra

$$\mathcal{U}_{[r^k]}(D_{2m+1}) \cong D\left(\frac{D_{2m+1}}{\langle r^k \rangle}\right) \cong D(D_{\mathrm{gcd}(k,2m+1)})\,. \tag{10.17}$$

The kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ is isomorphic to $\mathbb{C}\langle r^k \rangle = \mathbb{C}\mathbb{Z}_x$. We will denote its representations by ρ_l (with $0 \leq l < x$). They are defined in the usual way, with $r^{\mathrm{gcd}(k,2m+1)}$ taken as the preferred generator. The restriction of the irreps of $\mathcal{T}_{[r^k]}$ to $\mathrm{LKer}(\Gamma)$ is given by

$$\begin{array}{lll} \Omega^E_{J_0} \equiv \rho_0 & \Omega^E_{\alpha_l} \equiv \rho_l \oplus \rho_{-l} & \Omega^S_{J_0} \equiv \mathrm{gcd}(k,2m+1)\rho_0 \\ \Omega^E_{J_1} \equiv \rho_0 & \Omega^{r^p}_{\beta_l} \equiv \rho_l \oplus \rho_{-l} & \Omega^S_{J_1} \equiv \mathrm{gcd}(k,2m+1)\rho_0 \\ & & \Omega^S_{\alpha_j} \equiv \mathrm{gcd}(k,2m+1)(\rho_j \oplus \rho_{-j}) \end{array} \tag{10.18}$$

Here, the indices on the ρ's should be read modulo x.

- Now we take the condensate state in the module $V^s_{\gamma_0}$. Since the minimal normal subgroup of D_{2m+1} that contains s is D_{2m+1} itself, this condensate breaks the magnetic part of $D(D_{2m+1})$ completely and we are left with just the electric group D_{2m+1}, that is, we have
$$\mathcal{T}_{[s]}(D_{2m+1}) \cong \mathbb{C}D_{2m+1}\,. \tag{10.19}$$
Thus, the irreps of $\mathcal{T}_{[s]}$ are just the irreps J_0, J_1 and $\alpha_1, \ldots, \alpha_m$ of D_{2m+1}. The decomposition of D_{2m+1}-irreps into these gauge group irreps is
$$\begin{array}{l} \Pi^e_{J_0} \equiv J_0 \\ \Pi^e_{J_1} \equiv J_1 \end{array} \quad \Pi^{r^k}_{\beta_l} \equiv \begin{cases} J_0 \oplus J_1 & (l=0) \\ \alpha_l & (1 \leq l \leq m) \\ \alpha_{2m+1-l} & (m+1 \leq l < 2m+1) \end{cases} \quad \begin{array}{l} \Pi^s_{\gamma_0} \equiv J_0 \oplus \bigoplus_j \alpha_j \\ \Pi^s_{\gamma_1} \equiv J_1 \oplus \bigoplus_j \alpha_j \end{array} \tag{10.20}$$
Again, we see that pure fluxes may be turned into particles which carry a charge with respect to the residual symmetry.

Of the irreps of $\mathcal{T}_{[s]}$, only the trivial representation J_0 is not confined. In other words, all non-confined excitations over this condensate are "color" singlets. This means that

$$\mathcal{U}_{[s]}(D_{2m+1}) \cong \mathbb{C}\{e\}\,. \tag{10.21}$$

The Hopf kernel of the associated map $\Gamma : \mathcal{T}_{[s]} \to \mathcal{U}_{[s]}$ is all of $\mathcal{T}_{[s]}$. Hence the "restriction" to $\mathrm{LKer}(\Gamma)$ is trivial; the walls are just labeled by the irreps of $\mathcal{T}_{[s]}$.

11. Condensates of pure magnetic flux

11.1 Symmetry breaking

We will now study symmetry breaking by a state with pure flux yg_Ay^{-1} in the conjugacy class $A \subset H$. The vector $\phi \in \Pi^A_1$ that corresponds to this state is given by $\phi(x) = 1_{yN_A}(x)$.

According to proposition 6, the residual symmetry algebra $\mathcal{T}_{yg_Ay^{-1}}(H)$ is spanned by the matrix elements of the set of irreps (ρ, g) of $D(H)^*$ which have the property that ϕ is an eigenstate of the action of g^{-1} with eigenvalue equal to $\frac{\chi_\rho(g_A)}{d_\rho}$. In the case at hand, it is clear that the only eigenvalue of the action of any element of H that may occur is the value 1. It follows that ρ must be such that $\chi_\rho(g_A) = d_\rho$ and hence such that g_A lies in the kernel of ρ. Given such a ρ, we can find the corresponding elements g by solving the equation $\phi(gx) = \phi(x)$. In this case,we have

$$1_{yN_A}(x) = 1_{yN_A}(gx) = 1_{g^{-1}yN_A}(x)\,. \tag{11.1}$$

Now the functions 1_{yN_A} and $1_{g^{-1}yN_A}$ are equal exactly if $g^{-1} \in yN_Ay^{-1}$, or equivalently, $g \in yN_Ay^{-1} = N_{yg_Ay^{-1}}$. Thus, the admissible irreps (ρ, g) are those for which g_A lies in the kernel of ρ and g commutes with the condensed flux yg_Ay^{-1}. Following the same arguments as in section 10.1, we see that the allowed ρ span exactly the space of functions on H that are constant on the cosets of the minimal normal subgroup K_A of H that contains the class A. Thus the residual symmetry algebra is the Hopf subalgebra $\mathcal{T}_{yg_Ay^{-1}}(H)$ of $D(H)$ defined by

$$\mathcal{T}_{yg_Ay^{-1}}(H) := \left\{F \in D(H) | \forall k \in K_A : F(xk, z) = F(x, z) 1_{N_{yg_Ay^{-1}}}(z)\right\}. \tag{11.2}$$

Clearly, $\mathcal{T}_{yg_Ay^{-1}} \cong F(H/K_A \times N_{yg_Ay^{-1}})$ as a vector space and we see that $\mathcal{T}_{yg_Ay^{-1}}$ is a transformation group algebra, with $N_{yg_Ay^{-1}}$ acting on K_A through conjugation. Thus we may again use theorem 1 to write down the irreps of $\mathcal{T}_{yg_Ay^{-1}}$. They are labeled by an $N_{yg_Ay^{-1}}$-orbit $\mathcal{O} \subset H/K_A$ and an irrep τ of the stabilizer $N_\mathcal{O}$ of a chosen element $g_\mathcal{O} \in \mathcal{O}$. The irrep labeled by $\mathcal{O}$ and τ will be denoted $\Omega^\mathcal{O}_\tau$. It acts on the Hilbert space $F_\tau(N_{yg_Ay^{-1}}, V_\tau)$ in the usual way:

$$\left(\Omega^\mathcal{O}_\tau(F)\phi\right)(x) := \int_{N_{yg_Ay^{-1}}} dz\, F(xg_\mathcal{O}x^{-1}, z)\, \phi(z^{-1}x)\,. \tag{11.3}$$

The character $\psi^\mathcal{O}_\tau$ of $\Omega^\mathcal{O}_\tau$ is given as a function on $H/K_A \times N_{yg_Ay^{-1}}$ by (cf. (4.15))

$$\psi^\mathcal{O}_\tau(\eta, n) = 1_{N_\eta}(h) 1_\mathcal{O}(\eta) \psi_\tau(x_\eta^{-1} h x_\eta)\,. \tag{11.4}$$

The characters χ^B_β of the irreps Π^B_β of $D(H)$, viewed as $\mathcal{T}_{yg_Ay^{-1}}(H)$-modules are given by

$$\chi^B_\beta(gK_A, n) = \sum_{k \in K_A} 1_{N_{gk}}(n) 1_A(gk) \chi_\beta(x_{gk}^{-1} n x_{gk})\,, \tag{11.5}$$

where $n \in N_{yg_Ay^{-1}}$.

11.2 Confinement

We want to find out which of the irreps Ω^B_β of $\mathcal{T}_{yg_Ay^{-1}}$ are confined and which are not. To keep things simple, we take the condensed flux yg_Ay^{-1} to be just g_A. This can be done without any real loss of generality, since g_A was chosen arbitrarily in A. Again, the non-confined irreps have to satisfy the requirements (7.7) and again, these reduce to (7.24) and (7.25) (with $K = K_A$ and $N = N_{g_A} = N_A$), since $\mathcal{T}_{g_A}$ is isomorphic to a transformation group algebra. In the case at hand, where $\phi = 1_{N_A}$, (7.24) reduces to the requirement that

$$(\forall x \in N_A, \forall \eta \in B) \quad |x_\eta g_{\mathcal{O}}^{-1} x_\eta^{-1} x K_A \cap N_A| = |x K_A \cap N_A| \,. \tag{11.6}$$

By definition of x_η and $g_{\mathcal{O}}$, we have $x_\eta g_{\mathcal{O}}^{-1} x_\eta K_A = \eta$. using this and multiplying the sets in the above equation with x^{-1} from the right, we see that it reduces to

$$(\forall \eta \in B) \quad |\eta K_A \cap N_A| = |K_A \cap N_A|. \tag{11.7}$$

We know that $g_A \in K_A \cap N_A$ and thus that, if the above requirement is to hold, $\eta K_A \cap N_A$ must be non-empty. But this implies that $\eta = n K_A$ for some $n \in N_A$. On the other hand, if this is the case, then the above equation is always satisfied. Hence, the orbits B which are not confined are those whose elements can be written in the form nK_A for some $n \in N_A$.

The condition (7.25) becomes

$$(\forall x \in N_A, \forall \eta \in B) \quad \beta(x_\eta^{-1} x g_A x^{-1} x_\eta) = I \,. \tag{11.8}$$

Since $x \in N_A$ and $x_\eta \in N_A$ for all η, this reduces further to yield the condition

$$\beta(g_A) = I \tag{11.9}$$

on β. Basically, this says that β must be trivial on the minimal normal subgroup of N_B that contains g_A.

Let us compare the solutions that we have found to the set of solutions that we had found already in proposition 9 in section 7.3. The latter set consists of all $\Omega_\tau^{\mathcal{O}}$ for which the orbit $\mathcal{O}$ is made up of cosets of the form nK_A (with $n \in N_A$) and for which τ is trivial on $K_A \cap N_A$. Thus, we see that we have not found any extra orbits $\mathcal{O}$, but, depending on H, A and $N_{\mathcal{O}}$, we may have found extra irreps τ of $N_{\mathcal{O}}$, since the minimal normal subgroup of $N_{\mathcal{O}}$ that contains g_A can be smaller than $K_A \cap N_A$.

Thus we come back to a point that we touched upon already in section 7.1, namely the fact that we are in doubt whether it is always possible to give the full set of solutions to (7.7) a well-defined spin and and a well-defined braiding. We do know that braiding and spin are well-defined for the set of solutions that we had already found in section 7.3, since these are in one to one correspondence with the irreps of the quantum double of $N_A/(K_A \cap N_A)$. Therefore, we expect that the non-confined symmetry algebra for the condensates treated in this section should be $D(N_A/(K_A \cap N_A))$.

If the unconfined algebra is $D(N_A/(K_A \cap N_A))$, then the walls that are created by $\mathcal{T}_{g_A}$-excitations can be classified by the left or right Hopf kernel of the map $\Gamma : \mathcal{T}_{g_A} \to D(N_A/(K_A \cap N_A))$. We will take the right kernel, as given in proposition 10. Corollary 3 tells us that this Hopf kernel is isomorphic as an algebra to the tensor product $F((H/K_A)/\overline{N_A}) \otimes \mathbb{C}(N_A \cap K_A)$, where $\overline{N_A}$ is the subgroup of H/K_A which consists of elements of the form nK_A, with $n \in N$. In fact, RKer(Γ) is spanned by the elements of $\mathcal{T}$ which are of the form $1_{hK_A N_A} \otimes \delta_g$ with $g \in N_A \cap K_A$. The irreps of RKer(Γ) are tensor products of an irrep $E_{[\zeta]}$ of $F((H/K_A)/\overline{N_A})$ and an irrep ρ_l of $N_A \cap K_A$. We will denote them $E_{[\zeta]} \otimes \rho_l$. Here, $[\zeta]$ is notation for the $\overline{N_A}$-coset of ζ in H/K_A. We have

$$E_{[\zeta]} \otimes \rho_l(1_{hK_AN_A} \otimes g) = \delta_{[\zeta],[hK_A]}\rho_l(g)\,. \tag{11.10}$$

The decomposition of the $\mathcal{T}_{g_A}$-irrep $\Omega^{\mathcal{O}}_{\tau}$ into RKer(Γ) irreps may be found using the formula (4.13) for the character $\psi^{\mathcal{O}}_{\tau}$. We have

$$\begin{aligned}
\psi^{\mathcal{O}}_{\tau}(1_{hK_AN_A} \otimes \delta_g) &= \int_{\mathcal{O}} d\zeta \int_{K_A\cap N_A} dn\, 1_{hK_AN_A}\left(x_\zeta \xi_{\mathcal{O}} x_\zeta^{-1}\right) \delta_g(x_\zeta n x_\zeta^{-1})\chi_\alpha(n) \\
&= \int_{\mathcal{O}} d\zeta\, 1_{hK_AN_A}\left(x_\zeta \xi_{\mathcal{O}} x_\zeta^{-1}\right) \chi_\alpha(x_\zeta^{-1} g x_\zeta) \\
&= \int_{\mathcal{O}} d\zeta\, \delta_{[\zeta],[hK_A]}\chi_\alpha\left(x_\zeta^{-1} g x_\zeta\right).
\end{aligned} \tag{11.11}$$

From this, we read off that $\Omega^{\mathcal{O}}_{\tau}$ is the sum over $\zeta \in \mathcal{O}$ of those $E_{[\zeta]} \otimes \rho_l$ for which ρ_l is related to one of the $N_A \cap K_A$-irreps contained in τ by conjugation with x_ζ^{-1}. Of course, there may be multiplicities in the decomposition, for example if the coset $[\zeta]$ is the same for several $\zeta \in \mathcal{O}$. Also, note that the non-confined irreps of $\mathcal{T}_{g_A}$ all correspond to the trivial irrep $E_{[K_A]} \otimes 1$, as they should.

11.3 Examples of pure flux condensates

Pure flux condensates whose flux is central in H are gauge invariant and examples may be found in section 10.3. Here we treat the case where the flux of the condensate is non-central, so that not only the magnetic part of the double, but also the electric group is broken.

11.3.1 $H = D_{2m+1}$

1. Suppose the condensed flux is $r^k \in D_{2m+1}$. In that case, the residual symmetry algebra is the transformation group algebra $F(D_{2m+1}/K_{r^k}) \tilde{\otimes} \mathbb{C}N_{r^k}$, where K_{r^k} is the minimal normal subgroup that contains r^k and where $N_{r^k} = \langle\, r \,\rangle \cong \mathbb{Z}_{2m+1}$ is the centralizer of r^k in D_{2m+1}. From section 10.3.2, we know that $K_{r^k} = \langle\, r^k \,\rangle = \langle\, r^{\gcd(k,2m+1)} \,\rangle \cong \mathbb{Z}_x$, where x is equal to $(2m+1)/\gcd(k, 2m+1)$. We also recall that $D_{2m+1}/\mathbb{Z}_x \cong D_{\gcd(k,2m+1)}$. Hence,

$$\mathcal{T}_{r^k}(D_{2m+1}) \cong F(D_{\gcd(k,2m+1)})\,\tilde{\otimes}\, \mathbb{C}\mathbb{Z}_{2m+1}\,. \tag{11.12}$$

   The $D_{\gcd(k,2m+1)}$ is generated by $R = r\mathbb{Z}_x$ and $S = s\mathbb{Z}_x$ and we will write E for its unit element $e\mathbb{Z}_x$. The orbits of the action of N_{r^k} are then $\{E\}, \{R\}, \ldots, \{R^{\gcd(k,2m+1)-1}\}$ and $\{S, SR, \ldots, SR^{\gcd(k,2m+1)-1}\}$. The stabilizer of the orbits with one element is of course $N_{r^k} \cong \mathbb{Z}_{2m+1}$, while the orbit of S has stabilizer $K_{r^k} \cong \mathbb{Z}_x$. It follows that the representations of $\mathcal{T}_{r^k}$ may be written $\Omega^{R^p}_{\beta_l}$ (with $0 \leq p < \gcd(k, 2m+1)$, $0 \leq l < 2m+1$) and $\Omega^{S}_{\beta_l}$ (with $0 \leq l < x$). We see that $\mathcal{T}_{r^k}$ has $(2m+1)\gcd(k, 2m+1)$ irreps of dimension 1 and x irreps of dimension $\gcd(k, 2m+1)$. The squares of the dimensions add to the dimension of $\mathcal{T}_{r^k}$, which is $2(2m+1)\mathrm{gcd}(k,2m+1)$. The decomposition of $D(D_{2m+1})$-irreps into $\mathcal{T}_{r^k}$-irreps is as follows:

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega^E_{\beta_0} & \Pi^{r^k}_{\beta_l} \equiv \Omega^{R^k}_{\beta_l} \oplus \Omega^{R^{-k}}_{\beta_{-l}} \\ \Pi^e_{J_1} \equiv \Omega^E_{\beta_0} & \Pi^s_{\gamma_0} \equiv \bigoplus_l \Omega^S_{\beta_l} \\ \Pi^e_{\alpha_l} \equiv \Omega^E_{\beta_l} \oplus \Omega^E_{\beta_{-l}} & \Pi^s_{\gamma_1} \equiv \bigoplus_l \Omega^S_{\beta_l}\,. \end{array} \tag{11.13}$$

The irreps of $\mathcal{T}_{r^k}$ which are not confined are the $\Omega^{R^k}_{\beta_l}$ for which $\beta_l(r^k) = 1$, or in other words, those for which l is a multiple of x. It follows that the unconfined representations are automatically in one to one correspondence with the irreps of

$$\mathcal{U}_{r^k}(D_{2m+1}) \cong D\left(N_{r^k}/(K_{r^k} \cap N_{r^k})\right) \cong D(\mathbb{Z}_{\mathrm{gcd}(k,2m+1)})\,. \tag{11.14}$$

The right Hopf kernel of the Hopf map $\Gamma : \mathcal{T}_{r^k} \to D(\mathbb{Z}_{\mathrm{gcd}(k,2m+1)})$ is isomorphic to $F(\mathbb{Z}_2) \otimes \mathbb{C}\mathbb{Z}_x$ and we may denote its representations as $E_{[E]} \otimes \rho_l$ and $E_{[S]} \otimes \rho_l$ (with $0 \leq l < x$). Here, $[E]$ and $[S]$ denote the $\overline{N_{r^k}}$-cosets of the K_{r^k}-cosets E and S and ρ_l denotes the l^{th} representation of $\mathbb{Z}_x$, defined in the usual way, with $r^{\mathrm{gcd}(k,2m+1)}$ taken as the preferred generator of $\mathbb{Z}_x$. The restriction of the irreps of $\mathcal{T}_{r^k}$ to $\mathrm{RKer}(\Gamma)$ is given by

$$\Omega^{R^k}_{\beta_l} \equiv E_{[E]} \otimes \rho_{l \bmod x} \qquad \Omega^S_{\beta_l} \equiv \mathrm{gcd}(k,2m+1) E_{[S]} \otimes \rho_{l \bmod x}\,. \tag{11.15}$$

2. Now suppose the condensate has flux $s \in D_{2m+1}$. The minimal normal subgroup of D_{2m+1} that contains s is D_{2m+1} itself and the normalizer N_s of s is just $\{e, s\} \cong \mathbb{Z}_2$. Hence, this condensate leaves us with the symmetry algebra

$$\mathcal{T}_s(D_{2m+1}) \cong F(\mathbb{Z}_2) \cong \mathbb{C}\mathbb{Z}_2\,. \tag{11.16}$$

The irreps of this $\mathbb{Z}_2$ may be labeled Ω_{J_0} and Ω_{J_1} and the decomposition of the irreps of $D(D_{2m+1})$ is then given by

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega_{J_0} & \Pi^{r^k}_{\beta_l} \equiv \Omega_{J_0} \oplus \Omega_{J_1} \\ \Pi^e_{J_1} \equiv \Omega_{J_1} & \Pi^s_{\gamma_0} \equiv (m+1)\Omega_{J_0} \oplus m\Omega_{J_1} \\ \Pi^e_{\alpha_l} \equiv \Omega_{J_0} \oplus \Omega_{J_1} & \Pi^s_{\gamma_1} \equiv m\Omega_{J_0} \oplus (m+1)\Omega_{J_1}\,. \end{array} \tag{11.17}$$

Since $J_1(s) = -1 \neq 1$, it follows that Ω_{J_1} is confined, so that the only non-confined irrep of $\mathcal{T}_s$ is the "color singlet" Ω_{J_0}. Hence

$$\mathcal{U}_s(D_{2m+1}) \cong \mathbb{C}\{e\} \tag{11.18}$$

and the corresponding Hopf kernel equals $\mathcal{T}_s$.

12. Dyonic condensates

Attempts to study dyonic condensates in the same generality as electric or magnetic condensates meet with some problems of a technical nature. For example, the residual algebra after symmetry breaking does not have to be a transformation group algebra of the kind we discussed in section 7.3 (see the second part of section 12.1 for an example). Therefore, we will only treat some examples with specific groups and condensate vectors here, in order to give an idea of what one may expect. In the process, we also complete our treatment of condensates in theories where the gauge group is an odd dihedral group.

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12.1 $H = \mathbb{Z}_n$

First of all, let us check which condensates satisfy the requirements of trivial spin and self-braiding that we gave in section 8. As before, we denote our favorite generator of $\mathbb{Z}_n$ as r and we denote the representations of this group, defined in the usual way, as α_l (with $0 \leq l < n$). The representations of $D(\mathbb{Z}_n)$ may then be written $\Pi^{r^k}_{\alpha_l}$. The spin factor $s^{r^k}_{\alpha_l}$ of $\Pi^{r^k}_{\alpha_l}$ is just $\exp(-2\pi i k l/n)$ and so the requirement of trivial spin selects those $\Pi^{r^k}_{\alpha_l}$ for which we have

$$kl = 0 \bmod n\,. \tag{12.1}$$

Thus, given k, the allowed l are those which are 0 modulo $n/\gcd(k,n)$ and given l, the allowed k are those which are 0 modulo $n/\gcd(l,n)$. From this, we see immediately that, if n is a prime, there will be no allowed dyonic condensates (either l or k has to be zero). We will thus assume from now on that n is composite. For abelian groups, the requirement of trivial self-braiding is automatically satisfied for states with trivial spin, so the $\Pi^{r^k}_{\alpha_l}$ with $kl = 0 \bmod n$ all give good condensates.

To find the residual algebra $\mathcal{T}^k_l$ for a $\Pi^{r^k}_{\alpha_l}$-condensate, we have to find the representations of $D(\mathbb{Z}_n)^*$ that satisfy equation (6.13). Since $D(\mathbb{Z}_n)^* \cong \mathbb{C}\mathbb{Z}_n \otimes F(\mathbb{Z}_n)$, its representations may be labeled by an irrep α_q of $\mathbb{Z}_n$ and an element r^p of $\mathbb{Z}_n$. The equation (6.13) then selects those (α_p, r^q) for which $\alpha_p(r^k)\alpha_l(r^q) = 1$, or more explicitly, those (α_p, r^q) for which $\exp(2\pi i(kp + lq)/n) = 1$. This means that, to find the allowed p and q, we have to solve the equation

$$kp + lq = 0 \bmod n\,. \tag{12.2}$$

Rather than looking at the general solution of this equation for all k and l, we will examine two illustrative special cases:

1. First, let us take (n,k,l) such that $n = kl$ and $\gcd(k,l) = 1$. In this situation, we can easily find the solution to equation (12.2). Since $kp + lq$ is a multiple of n, say mn, we may solve for p to get

$$p = \frac{1}{k}(mn - lq) = lm - \frac{l}{k}q\,, \tag{12.3}$$

using $n = kl$ in the second equality. Since p has to be integer, it follows that $\frac{l}{k}q$ must be an integer. Since $\gcd(k,l) = 1$, the fraction $\frac{l}{k}$ is irreducible and hence $\frac{l}{k}q$ can only be integer if q is a multiple of k. But then it follows from the equation above that p is a multiple of l. On the other hand, it is clear from $n = kl$ that any (p,q) for which p is an l-fold and q is a k-fold will solve (12.2). Thus the residual algebra $\mathcal{T}^k_l$ is spanned by the (matrix elements of) the representations (α_p, r^q) for which $p = 0 \bmod l$ and $q = 0 \bmod k$. Now since $n = kl$, the irreps α_p of $\mathbb{Z}_n$ with $p = 0 \bmod l$ correspond exactly to the irreps of the quotient group $\mathbb{Z}_n/\langle\, r^k\,\rangle \cong \mathbb{Z}_k$. Hence,

$$\mathcal{T}^k_l \cong F(\mathbb{Z}_n/\langle\, r^k\,\rangle)\,\tilde{\otimes}\,\mathbb{C}\langle\, r^k\,\rangle \cong F(\mathbb{Z}_k)\,\tilde{\otimes}\,\mathbb{C}\mathbb{Z}_l\,. \tag{12.4}$$

We see that $\mathcal{T}^k_l$ is a transformation group algebra of the kind treated in section 7.3, where both the normal subgroup K and the subgroup N of these sections equal

$\langle r^k \rangle \cong \mathbb{Z}_l$ in this case. The representations of $\mathcal{T}_l^k$ may thus be denoted Ω_s^r, with $0 \le r < k$ and $0 \le s < l$ and the restriction of the irreps of $D(\mathbb{Z}_n)$ to $\mathcal{T}_l^k$ is given by

$$\Pi^{r^a}_{\alpha_b} \equiv \Omega^{a \bmod k}_{b \bmod l} \, . \tag{12.5}$$

Using the theory of section 7.3, one may see that all the Ω_s^r with $(r,s) \neq (0,0)$ are confined. The unconfined algebra $\mathcal{U}_l^k$ is thus the group algebra of the trivial group and the Hopf kernel of the Hopf map $\Gamma : \mathcal{T}_l^k \to \mathcal{U}_l^k$ is all of $\mathcal{T}_l^k$, implying that walls and $\mathcal{T}_l^k$-particles are in one-to-one correspondence.

2. Now consider the case where $l = -k \bmod n$. Equation (12.2) then becomes

$$k(p-q) = 0 \bmod n \tag{12.6}$$

so that the allowed (p,q) are those for which $p = q \bmod n/\mathrm{gcd}(k,n)$. It follows that

$$\mathcal{T}_l^k \cong \mathbb{C}(\mathbb{Z}_n \times \mathbb{Z}_{\mathrm{gcd}(k,n)}) \, , \tag{12.7}$$

where (α_1, r) generates the $\mathbb{Z}_n$ and where either $(\alpha_{n/\mathrm{gcd}(k,n)}, e)$ or $(\alpha_0, r^{n/\mathrm{gcd}(k,n)})$ can be taken as the generator for the $\mathbb{Z}_{\mathrm{gcd}(k,n)}$. We will take the latter possibility. One should notice that, in contrast to everything we have seen up to now, the full residual algebra is not generated by the residual magnetic and the residual electric symmetry algebra. The residual electric and magnetic algebra are generated by $(\alpha_0, r^{n/\mathrm{gcd}(k,n)})$ and $(\alpha_{n/\mathrm{gcd}(k,n)}, e)$ respectively and are both isomorphic to $\mathbb{C}\mathbb{Z}_{\mathrm{gcd}(k,n)} \cong F(\mathbb{Z}_{\mathrm{gcd}(k,n)})$. The total residual algebra is $\mathbb{C}(\mathbb{Z}_n \times \mathbb{Z}_{\mathrm{gcd}(k,n)})$ and contains for example the element (α_1, r), which cannot be generated from the elements of the residual electric and magnetic algebras. Clearly, the residual algebra is not a transformation group algebra of the kind treated in section 7.3. This phenomenon is not limited to abelian H, but can also occur for non-abelian H. In fact, one may check that it does so already for some condensates in a $D(D_4)$-theory.

The representations of $\mathcal{T}_l^k$ may be written $\chi_{a,b}$, with $0 \le a < n$, $0 \le b < \mathrm{gcd}(k,n)$. They are defined in the usual way, through

$$\begin{aligned} \chi_{a,b}(\alpha_1, r) &= e^{2\pi i a/n} \\ \chi_{a,b}(\alpha_0, r^{n/\mathrm{gcd}(k,n)}) &= e^{2\pi i b/\mathrm{gcd}(n,k)} \, . \end{aligned} \tag{12.8}$$

On the magnetic part of $\mathcal{T}_l^k$, $\chi_{a,b}$ is given by $\chi_{a,b}(\alpha_{n/\mathrm{gcd}(k,n)}, e) = e^{2\pi i(a-b)/\mathrm{gcd}(n,k)}$, as follows from the definition above. The restriction of the irreps of $D(\mathbb{Z}_n)$ to T is given by

$$\Pi^{r^p}_{\alpha_q} \equiv \chi_{p+q,\, q} \, , \tag{12.9}$$

where the second q on the right hand side should be read modulo $\mathrm{gcd}(k,n)$.

Since we cannot apply the theory of section 7.3 here, we have to refer back to the requirements (7.7) in section 7.1 in order to determine which of the representations of $\mathcal{T}_l^k$

are confined and which are not. After some algebra, the first of these requirements, applied to $f = \chi_{a,b}$, reduces to

$$e^{-2\pi i(a-b)k/n} = 1\,, \qquad (12.10)$$

from which it follows that

$$a = b \bmod \frac{n}{\gcd(n,k)}\,. \qquad (12.11)$$

Note that $n/\gcd(n,k)$ is a divisor of both n and $\gcd(n,k)$, since $k^2 = 0 \bmod n$, using (12.1). As a consequence, the above equation retains its usual meaning, despite the fact that a is only defined modulo n and b is only defined modulo $\gcd(n,k)$.

The second requirement in (7.7), applied to $f = \chi_{a,b}$, becomes

$$e^{2\pi i bk/n} = 1\,, \qquad (12.12)$$

so that we have

$$b = 0 \bmod \frac{n}{\gcd(n,k)}\,. \qquad (12.13)$$

Hence, the unconfined representations of $\mathcal{T}_l^k$ are just those $\chi_{a,b}$ for which both a and b are multiples of $n/\gcd(n,k)$. All in all, this leaves $\gcd(n,k)$ possibilities for a and $(\gcd(n,k))^2/n$ possibilities for b, so that the non-confined algebra $\mathcal{U}_l^k$ is given by

$$\mathcal{U}_l^k \cong \mathbb{C}(\mathbb{Z}_{\gcd(n,k)} \times \mathbb{Z}_{(\gcd(n,k))^2/n})\,. \qquad (12.14)$$

As an example, consider the case of $D(\mathbb{Z}_9)$, with a condensate given by $k = -l = 3$. The only non-confined irreps of $\mathcal{T}_{-3}^3$ are then $\chi_{0,0}, \chi_{3,0}$ and $\chi_{6,0}$ and the unconfined algebra is $\mathbb{C}\mathbb{Z}_3$. We give a graphical representation of our results for this case in figure 1.

12.2 $H = D_{2m+1}$

In this section, we complete our treatment of condensates in the odd dihedral gauge theories. First, we find out which states in dyonic representations of $D(D_{2m+1})$ satisfy the conditions of trivial spin and trivial self-braiding. From table 2, we read off that the only dyonic irreps of $D(D_{2m+1})$ which have trivial spin are those $\Pi_{\beta_l}^{r^k}$ for which $\exp(2\pi ikl/(2m+1)) = 1$, or in other words, for which $kl = 0 \bmod 2\mathrm{m}+1$. It follows that there are no admissible dyonic condensates when $2m+1$ is prime. If $2m+1$ is not a prime, then there will be dyons with trivial spin and one may check easily that any state in the module of one of the $\Pi_{\beta_l}^{r^k}$ with $kl = 0 \bmod 2\mathrm{m}+1$ also has trivial self-braiding. Therefore, all states in the modules of these dyonic irreps may in principle be condensed.

Now suppose that we have condensed a state ϕ in the module of $\Pi_{\beta_l}^{r^k}$. To find the residual symmetry algebra, we have to solve equation (6.12). The representations (ρ, g)

	$\Pi_{J_0}^e$	$\Pi_{J_1}^e$	$\Pi_{\alpha_j}^e$	$\Pi_{\beta_l}^{r^k}$	Π_1^s	Π_γ^s
d_α^A	1	1	2	2	$2m+1$	$2m+1$
s_α^A	1	1	1	q^{-kl}	1	-1

Table 2: Dimensions and spin factors for the irreps of $D(D_{2m+1})$.

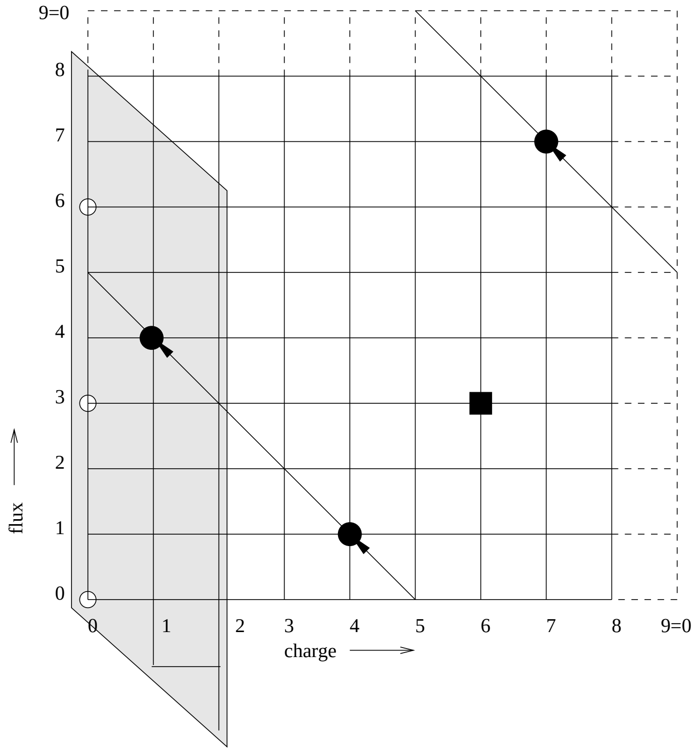

Figure 1: Flux-charge lattice for a $D(\mathbb{Z}_9)$ theory. We assume that a condensate of particles with flux $r^3 \equiv 3$ and charge $\alpha_{-3} \equiv -3 \equiv 6$ forms. The condensed irrep is indicated as a square dot. The residual symmetry algebra $\mathcal{T}^3_{-3}$ is a group algebra $\mathbb{C}(\mathbb{Z}_9 \times \mathbb{Z}_3)$. Two $D(H)$-irreps in the lattice are equivalent as $\mathcal{T}^3_{-3}$-irreps if one can be reached from the other through translations by the "condensate vector" $(-3, 3)$. This way, $D(H)$-irreps are identified in trios. The trio in the picture corresponds to the $\mathcal{T}$-irrep $\chi_{5,1}$. The shaded region contains one representative from each trio and is thus a diagram of all $\mathcal{T}$-irreps. The small white circles indicate the three unconfined irreps of $\mathcal{T}$, which correspond to the irreps of $\mathcal{U} \cong \mathbb{C}\mathbb{Z}_3$.

of $D(D_{2m+1})^*$ which satisfy this equation are those for which $\frac{\chi_\rho(r^k)}{d_\rho}$ is a root of unity and ϕ is an eigenvector of g^{-1} with eigenvalue equal to this root of unity. Thus, let us first find all irreps ρ of D_{2m+1} for which $\frac{\chi_\rho(r^k)}{d_\rho}$ is a root of unity. From table 1 one may read off that these are J_0, J_1 and those α_j for which $2\cos(2\pi jk/(2m+1)) = 2$, or equivalently $jk = 0 \bmod 2m+1$. This leaves exactly those j which are multiples of $x_k := (2m+1)/\gcd(k, 2m+1)$. In all these cases, $\frac{\chi_\rho(r^k)}{d_\rho}$ actually equals 1, or equivalently, ρ is trivial on r^k. It follows that the allowed ρ correspond to the irreps of the quotient group $D_{2m+1}/\langle r^k \rangle$, which is isomorphic to $D_{\gcd(k,2m+1)}$. The residual symmetry algebra will now be spanned by the (ρ, g) with ρ in the set we have just found and g an element of the subgroup of D_{2m+1} that leaves ϕ invariant. This subgroup will depend on ϕ. Therefore, let us write ϕ more explicitly as $a\phi_{r^k} + b\phi_{r^{-k}}$. Here ϕ_{r^k} and $\phi_{r^{-k}}$ are just the two basis functions for $V^{r^k}_{\beta_l}$ as defined through (4.11). We have dropped the index i in (4.11), since the module of β_l is one-dimensional. Using the formula (4.12) for the matrix elements of $\Pi^{r^k}_{\beta_l}$ with respect to this basis, we can now write the action of the elements of D_{2m+1} on ϕ

explicitly as

$$\begin{aligned}\Pi^{r^k}_{\beta_l}(1\otimes\delta_{r^p})(a\phi_{r^k}+b\phi_{r^{-k}}) &= e^{\frac{2\pi ilp}{2m+1}}a\phi_{r^k}+e^{\frac{-2\pi ilp}{2m+1}}b\phi_{r^{-k}}\\ \Pi^{r^k}_{\beta_l}(1\otimes sr^p)(a\phi_{r^k}+b\phi_{r^{-k}}) &= e^{\frac{-2\pi ilp}{2m+1}}b\phi_{r^k}+e^{\frac{2\pi ilp}{2m+1}}a\phi_{r^{-k}}\,.\end{aligned} \tag{12.15}$$

From the first of these equations, we see that, independently of the choice of (a,b), r^p will leave ϕ invariant only if $\exp(2\pi ilp/(2m+1))=1$. In other words, p has to be a multiple of $x_l := (2m+1)/\gcd(2m+1,l)$. From the second equation above, we see that sr^p will leave ϕ invariant only if $b=\exp(2\pi ilp/(2m+1))a$. If no such relation between a and b exists, then none of the elements $sr^p\in D_{2m+1}$ will leave ϕ invariant and the subgroup of D_{2m+1} that does leave ϕ invariant is just the $\mathbb{Z}_{\gcd(2m+1,l)}$ generated by r^{x_l}. If we do have $b=\exp(2\pi ilp/(2m+1))a$ for some p, then the required subgroup of D_{2m+1} is the $D_{\gcd(2m+1,l)}$ generated by r^{x_l} and sr^p. All these $D_{\gcd(2m+1,l)}$ subgroups actually represent the same physics, since they are conjugates in D_{2m+1} (or equivalently, the corresponding condensates are all related by gauge transformations). We have thus found two distinct possibilities for the residual symmetry algebra $\mathcal{T}$, which we will call $\mathcal{T}^k_l$ and $\bar{\mathcal{T}}^k_l$. These algebras are given by

$$\begin{aligned}\mathcal{T}^k_l &\cong F\left(D_{2m+1}/\langle\, r^k\,\rangle\right)\tilde{\otimes}\,\mathbb{C}\langle\, r^{x_l}\,\rangle \cong F(D_{\gcd(2m+1,k)})\,\tilde{\otimes}\,\mathbb{C}\mathbb{Z}_{\gcd(2m+1,l)}\\ \bar{\mathcal{T}}^k_l &\cong F\left(D_{2m+1}/\langle\, r^k\,\rangle\right)\tilde{\otimes}\,\mathbb{C}\langle\, r^{x_l}, s\,\rangle \cong F(D_{\gcd(2m+1,k)})\,\tilde{\otimes}\,\mathbb{C}D_{\gcd(2m+1,l)}\,.\end{aligned} \tag{12.16}$$

Both $\mathcal{T}^k_l$ and $\bar{\mathcal{T}}^k_l$ are thus transformation group algebras of the form $F(H/K)\,\tilde{\otimes}\,\mathbb{C}N$. For $\mathcal{T}^k_l$, we have $K=\langle\, r^k\,\rangle$ and $N=\langle\, r^{x_l}\,\rangle$, whereas for $\bar{\mathcal{T}}^k_l$, we have $K=\langle\, r^k\,\rangle$ and $N=\langle\, r^{x_l}, s\,\rangle$. Because of this, the decomposition of $D(D_{2m+1})$-irreps into $\mathcal{T}$-irreps proceeds in the same way as for the electric and magnetic cases. Also, the theory of section 7.3 may be applied to treat confinement. One finds that the unconfined algebras $\mathcal{U}^k_l$ and $\bar{\mathcal{U}}^k_l$ are isomorphic to the quantum doubles of the groups $N/(K\cap N)$. Now it turns out that we have $K\subset\langle\, r^{x_l}\,\rangle$ and hence $K\subset N$ in both cases. To see this, remember that we have $kl=0 \bmod 2\mathrm{m}+1$ and thus $\gcd(k,2m+1)\gcd(l,2m+1)=0 \bmod 2\mathrm{m}+1$. Hence,

$$\gcd(k,2m+1)\gcd(l,2m+1)=q\,(2m+1) \tag{12.17}$$

for some integer q and it follows that $\gcd(k,2m+1)=qx_l$ and $\langle\, r^k\,\rangle\subset\langle\, r^{x_l}\,\rangle$ (note that $\langle\, r^k\,\rangle=\langle\, r^{\gcd(k,2m+1)}\,\rangle$). The integer q has the property that it divides both $\gcd(k,2m+1)$ and $\gcd(l,2m+1)$. Using this, one now sees easily that

$$\begin{aligned}\mathcal{U}^k_l &\cong D(\mathbb{Z}_q)\\ \bar{\mathcal{U}}^k_l &\cong D(D_q)\,.\end{aligned} \tag{12.18}$$

When $q=1$, this means that there is full confinement of $\mathcal{T}^k_l$-irreps, while on the other hand, there are still four unconfined $\bar{\mathcal{T}}^k_l$-irreps, since $\bar{\mathcal{U}}^k_l\cong D(\mathbb{Z}_2)$.

The Hopf kernels of the maps $\Gamma:\mathcal{T}^k_l\to\mathcal{U}^k_l$ and $\bar{\Gamma}:\bar{\mathcal{T}}^k_l\to\bar{\mathcal{U}}^k_l$ can also be determined, following the treatment in section 7.3. We find that

$$\begin{aligned}\mathrm{RKer}(\Gamma) &\cong F(D_{x_l})\otimes\mathbb{C}\mathbb{Z}_{x_k}\\ \mathrm{RKer}(\bar{\Gamma}) &\cong F\left(D_{\gcd(k,2m+1)}/D_{\gcd(k,2m+1)/x_l}\right)\otimes\mathbb{C}\mathbb{Z}_{x_k}.\end{aligned} \tag{12.19}$$

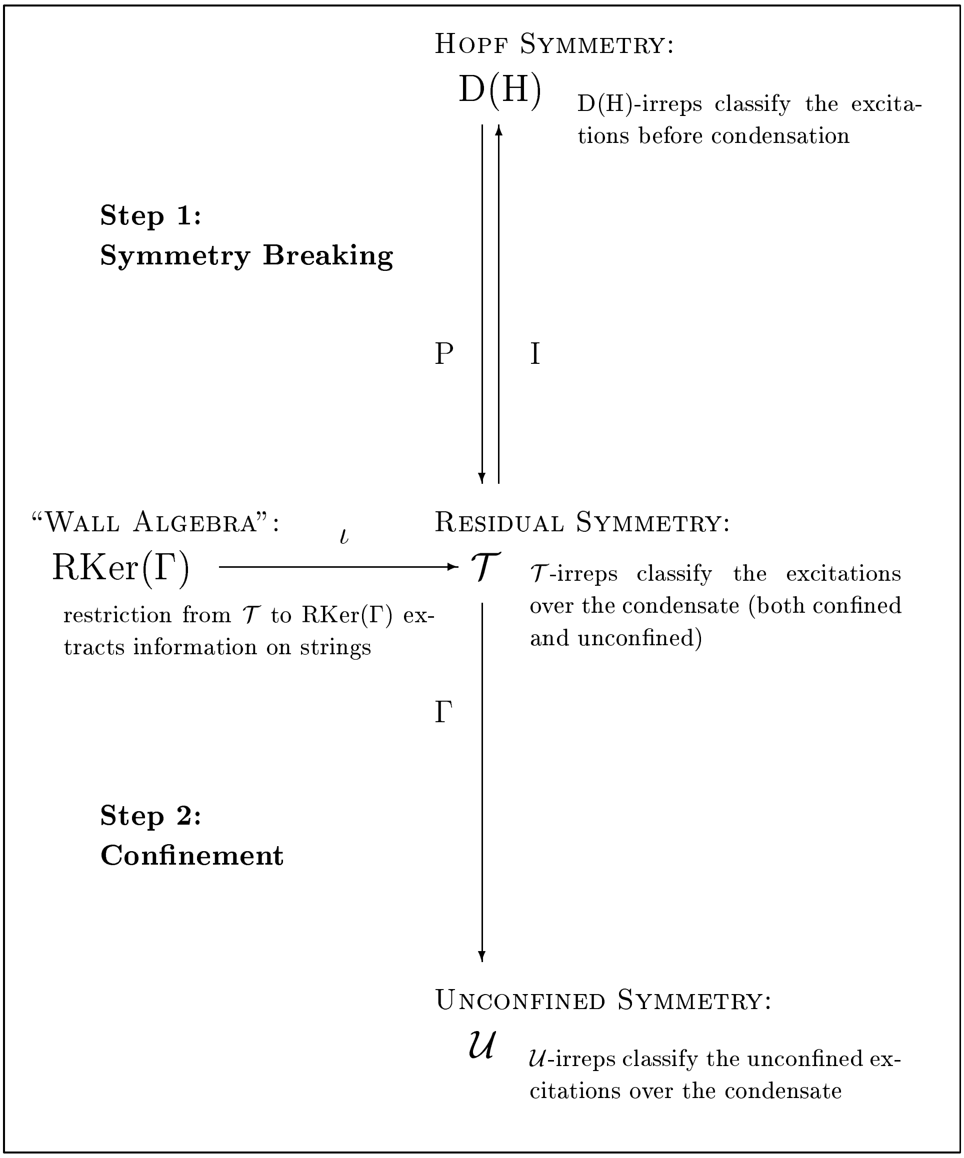

Figure 2: Schematic picture of the structures that play a role in this paper.

13. Summary and outlook

The general picture that emerges from this paper can be seen in figure 2. In words, it is as follows. The formation of a condensate induces symmetry breaking from $D(H)$ to the Hopf subalgebra $\mathcal{T} \subset D(H)$ which is the Hopf stabilizer of the condensate state. The ensuing confinement is described by a Hopf projection Γ of $\mathcal{T}$ onto an "unconfined" symmetry algebra $\mathcal{U}$, whose irreps label the free charges over the condensate. Walls or strings in the condensate are labeled by the restrictions of $\mathcal{T}$-irreps to the right Hopf kernel of Γ. In the diagram, I denotes the (Hopf) inclusion of $\mathcal{T}$ into $D(H)$, ι denotes the inclusion of LKer(Γ) into $\mathcal{T}$ and P denotes the orthogonal projection of $D(H)$ onto $\mathcal{T}$, which we use in our definition of $\mathcal{U}$. To the information in the diagram, we should add that all "baryonic" excitations on the condensate can be constructed by fusing together a number of confined particles, labeled by $\mathcal{T}$-irreps, in such a way that the overall fusion product has a non-confined charge, labeled by a $\mathcal{U}$-irrep.

Note that the role that the unconfined algebra $\mathcal{U}$ plays in the $D(H)$-theory is quite comparable to the role that $D(H)$ plays in the gauge theory with continuous gauge group G of which our discrete gauge theory is a Higgsed version. Just like $D(H)$ classifies the free excitations over the Higgs condensate in the continuous gauge theory, $\mathcal{U}$ classifies the free excitations over the condensate in the $D(H)$-theory. In fact, the different unconfined algebras we have found for specific condensates are typically themselves quantum doubles of a group related to H. For example:

- For purely electric condensates, we have found that $\mathcal{U}$ is the quantum double of the stabilizer N of the condensate in H. This is just what we expected, since the only effect of condensing one of the electric particles of the $D(H)$-theory is to modify the electric condensate of the G-theory in such a way that the residual gauge group is now N rather than H.

- For gauge invariant magnetic condensates, we have found that $\mathcal{U}$ is the quantum double of the quotient group H/K, where K is the group generated by the elements of the conjugacy class that labels the condensate. This is also in accordance with the intuition, since the division by K can be seen as a consequence of the fact that, after condensation, the flux of any particle can only be determined up to the condensed flux.

In a sense, we can describe the condensed phases of the $D(H)$-theory even better than the $D(H)$-theory itself describes the Higgs phase of the G-theory, since the algebra $\mathcal{T}$ that we obtain after symmetry breaking gives us information on the possible substructures of the free excitations over the condensate.

Nevertheless, there is still much work to be done. First of all, from the requirements (7.7) that we found in section 7, it is not clear that the set of irreps of $\mathcal{U}$ will always have a well-defined braiding. Although this does happen in the examples with electric and gauge invariant magnetic condensates (where $\mathcal{U}$ is even quasitriangular), we do not expect that the equations (7.7) will guarantee this in general. Therefore, we expect that supplementary conditions will be necessary for a completely satisfactory definition of $\mathcal{U}$. Secondly, it would be good to have some "independent" theoretical confirmation of the results in this paper. One could for example try to find the phases that we are predicting in numerical calculations on a lattice. It is also important to generalize the techniques for the breaking of Hopf algebra symmetries that we have developed in this paper, both to the case where the symmetry algebra is infinite dimensional and to the case where it is no longer a Hopf algebra, but only a quasi-Hopf algebra or even a weak quasi-Hopf algebra or Hopf algebroid. This would extend the applicability of our symmetry breaking scheme enormously. A theory of symmetry breaking for infinite dimensional Hopf algebras could for example have interesting applications in the study of (2+1)-dimensional gravity using the quantum group theoretical framework of [22, 23]. A generalization to weak quasi-Hopf algebras would bring any physical system which has a description in terms of Chern-Simons theory or rational conformal field theory within the reach of our methods. Finally, it would of course be extremely interesting if the treatment of symmetry breaking and coninement that we give here could be extended to gauge theories in $3+1$ (or higher) dimensions. One might begin to think of such an extension starting from the ideas presented in [47].

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